Prelim Solutions

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1 Notes

Complex analysis notes goes here! Some important tricks, formulas, theorems and Heuristics. To be written at the end.

Theorem 1 (Schwarz Lemma)**.** *Let $f : \mathbb{D} \to \mathbb{D}$ be analytic function. If $f(0) = 0$, then $|f(z)| \leq |z|$ for every $z \in \mathbb{D}$. Moreover, the equality is achieved if and only if $f(z) = \lambda z$ for some λ such that $|\lambda| = 1$.*

Proof. Let $\mathbb{D}_r$ be the disk of radius r centered at the origin. Define

$$g(z) = \begin{cases} \frac{f(z)}{z}, & z \neq 0 \\ f'(0), & z = 0 \end{cases}.$$

It is clear that g is analytic on disk. By maximum modulus principle, we obtain that for any $z \in \mathbb{D}_r$, we have $|g(z)| \leq \frac{1}{r}$. Taking the limit as $r \to 1$, we obtain that $|g(z)| \leq 1$ for every $z \in \mathbb{D}$.

Suppose that the equality $|f(z)| = z$ holds. It follows that $|f'(0)| = 1$. Applying maximum modulus principle on g yields that g is constant. Therefore, $g(z) = f'(0)$ for all z. In other words, $f(z) = f'(0)z$ for all $z \in \mathbb{D}$. $\square$

Recall that given $z \in \mathbb{D}$, there is an automorphism ϕ_z of $\mathbb{D}$ that sends 0 to z. In particular, given a function $f : \mathbb{D} \to \mathbb{D}$ and $z \in \mathbb{D}$, the map $F : \mathbb{D} \to \mathbb{D}$ defined by

$$F(w) = \phi^{-1}_{f(z)} \circ f \circ \phi_z(w)$$

satisfies $F(0) = 0$. Therefore, we obtain $|F(w)| \leq |w|$. Expanding the definition of F, we obtain the following inequality, called **Pick's inequality**

$$\left\| \frac{f(z) - f(w)}{1 - \overline{f(z)}f(w)} \right\| \leq \left\| \frac{z - w}{1 - \overline{z}w} \right\|, \quad \text{for all } z, w \in \mathbb{D}. \tag{1}$$

A straightforward corollary of the Pick's inequality is the following inequality

$$\frac{|f'(z)|}{1 - |f(z)|^2} \leq \frac{1}{1 - |z|^2}. \tag{2}$$

Of course one can write a analogous statements for upper half plane. Add it someday.

More about it!
Note how Pick's inequality 'disallows' certain holomorphic functions. For example, see Problem 5 in 1, there does not exist an analytic function $f : \mathbb{D} \to \mathbb{D}$ such that $f(\frac{1}{2}) = 2/3$ and $f(\frac{1}{4}) = \frac{1}{3}$. Contrast this with the fact that given any $z_1 \neq z_2, w_1, w_2 \in \mathbb{D}$, one can always construct a smooth function $f : \mathbb{D} \to \mathbb{D}$ that interpolates (z_i, w_i), that is, $f(z_i) = w_i$. The conclusion to draw is that the interpolation problem in complex variables is more interesting. It is natural to ask if Pick's inequality is sufficient for the existence of a function f which interpolates $(z_i, w_i), i = 1, 2$. The answer is 'Yes' and the proof is easy! (Left as an exercise)!

In fact, one can ask more generally the following question. Let $(z_i, w_i), i \in \{1, \ldots, n\}$ such that $|z_i| < 1, |w_i| \leq 1$ be given data. What is a necessary and sufficient condition for the existence of a holomorphic function $f : \mathbb{D} \to \mathbb{D}$ such that $f(z_i) = w_i$? This problem is a classical problem known as **Pick-Nevanlinna interpolation problem**. Of course, the problem lends itself to obvious generalizations in higher dimensions and operator-valued maps. Despite its irresistible beauty, we will not pursue its generalizations here. But it would be a sin to not give an answer to the Pick-Nevanlinna interpolation problem.

Theorem 2 (Donald Marshall, Michigan Math. J. 21 (1974))**.** *Let $(z_1, \ldots, z_n)$ be an n-tuple of points in the unit disk. Let $(w_1, \ldots, w_n)$ be an n-tuple of complex numbers. Let M be the $n \times n$ matrix defined by*

$$M_{ij} = \frac{1 - w_i \overline{w_j}}{1 - z_i \overline{z_j}}.$$

The matrix M is positive semi-definite if and only if there exists a holomorphic function $f : \mathbb{D} \to \mathbb{D}$ such that $f(z_i) = w_i$ for all i. Moreover, the interpolating function f is unique if and only if $\det(M) = 0$.

Proof. Add Proof! □

Problem 1
Find a conformal map from the half strip $A = \{z = x + iy \in \mathbb{C} : x > 0, 0 < y < 1\}$.

Solution: The map $\phi(z) = e^{-\pi z}$ takes the strip A onto $\{z \in \mathbb{C} : |z| < 1, \Im(z) > 0\}$. Let $T(z) = \frac{z-1}{z+1}$. Then, $(T \circ \phi(z))^2$ works.

Problem 3
Let $\Omega \subseteq \mathbb{C}$ be a domain and K be a compact subset of Ω. Let $\mathcal{F}$ be the family of bounded holomorphic functions on Ω which have at least one zero in K. Show that there exists a constant $C < 1$, such that $||f||_K \leq C||f||_\Omega$.

Solution: Observe that the inequality holds trivially if $f \equiv 0$, and if f is not identically zero, then by dividing f by $||f||_\Omega$, it suffices to show the inequality only when $||f|| \leq 1$. Let $\mathcal{F}_1$ be the family of holomorphic functions on Ω such that $||f||_\Omega \leq 1$ and such that f has at least one zero in K. The inequality $||f||_K \leq C||f||_\Omega$ holds trivially with the constant $C = 1$. Let

$$C_0 := \inf\{C : ||f||_K \leq ||f||_\Omega\} \text{for all} f \in \mathcal{F}_1.$$

We wish to show that $C_0 < 1$. If not, then there exists a sequence of functions $f_n \in \mathcal{F}_1$ such that $||f_n||_K \geq (1 - \frac{1}{n})||f_n||_\Omega$. Since $\mathcal{F}_1$ is uniformly bounded, it is normal by the Montel's theorem. In particular, there exists a subseqeunce of f_n that converges uniformly on compact subsets of Ω to a holomorphic function f. The set of zeroes of the subsequence of f_n is contained in the compact set K, therefore, passing to a further subsequence and relabelling, we may assume that $f_n(z_n) = 0$ and $z_n \to z_0 \in K$. Therefore, $f(z_0) = 0$. In particular, $f \in \mathcal{F}_1$. And, it is clear that $||f||_K = ||f||_\Omega$. But f must achieve $||f||_K$ on K, and therefore, by the maximum modulus principle, we obtain that f is a constant. Since f has a zero in K, it follows that f must be identically zero.

Problem 5
Let A be a subset of positive, finite measure in $\mathbb{R}$. Let $f(x) = \chi_A \star \chi_A$. Show that the support of f contains an open set.

Solution: We first observe that $f(x) \geq 0$, and by Tonelli's theorem one we obtain that $\int_{\mathbb{R}} f(x)dx = \mu(A)^2 > 0$. It follows that f must be positive on a set of positive measure. We will show that f is continuous, and therefore, it will follow that $\{f > 0\}$ is a non-empty open set.

To see that f is continuous, we note that $\chi_A(x - y) \to \chi_A(x_0 - y)$ almost everywhere as $x \to x_0$. It follows by DCT that

$$|f(x_1) - f(x_2)| \leq \int_A |\chi_A(x_1 - y) - \chi_A(x_2 - y)|dy \to 0,$$

as $|x_1 - x_2| \to 0$.

Problem 6
Let $1 \leq p < q < \infty$. Prove that $L^q \not\subseteq L^p$ if and only if X contains set of arbitrarily large measure.

Solution: Suppose X has finite measure. Let $f \in L^q$, then applying Hölder's inequality, we obtain that

$$\int_X |f|^p \leq (\int_X |f|^q)^{p/q} (\mu(X))^{1-p/q} < \infty. \tag{3}$$

That is, $L^q \subseteq L^p$.

Suppose X contains sets of arbitrarily large measure. In particular, there are disjoint subsets A_n such that $2^n \leq \mu(A_n) < \infty$. Now let $f := \sum_{n=1}^{\infty} \mu(A_n)^{-1/p} \chi_{A_n}$. For $q > p$, we observe that

$$||f||_q^q = \leq \sum_{n=1}^{\infty} \mu(A_n)^{1-q/p} \leq \sum_{n=1}^{\infty} 2^{i(1-q/r)} < \infty,$$

because $1 - q/r < 0$. On the other hand,

$$\int |f|^p = \sum_{n=1}^{\infty} \mu(A_n)^{-1} \mu(A_n) = \sum_{n=1}^{\infty} 1 = \infty.$$

Problem 8

Let $(X, ||\cdot||)$ be a real Banach space. For $z \in X$, define

$$U_z = \{f \in X^* : ||f|| = ||z||, f(z) = ||z||^2\}.$$

Show that U_z is non-empty, closed, convex for each $z \in X$.

Solution: Let $f, g \in U_z$ and $\lambda \in (0,1)$. We note that $||\lambda f + (1-\lambda)g|| \leq ||z||$ and

$$\lambda f(z) + (1-\lambda)g(z) = ||z||^2.$$

It follows that $||\lambda f + (1-\lambda)g|| = ||z||$ and $\lambda f + (1-\lambda)g(z) = ||z||^2$. Hence, U_z is convex.

To see that U_z is closed, let f_n be a sequence in U_z which converges to f in norm topology, then

$$-||f - f_n|| + ||z|| \leq ||f|| \leq ||f - f_n|| + ||f_n|| = ||f - f_n|| + ||z||.$$

Therefore, $||f|| = ||z||$ and it is clear that $f(z) = ||z||^2$ and hence U_z is closed.

Finally, let $X_z = z\mathbb{R} \subseteq X$. Define $f(z) = ||z||^2$ and linearly extend it on X_z by setting $f(\lambda z) = \lambda ||z||^2$. Note that $||f|| = ||z||$. We know by the Hahn-Banach theorem that f has a norm preserving extension $\tilde{f} \in X^*$. Clearly, $||\tilde{f}|| = ||z||$ and $\tilde{f}(z) = ||z||^2$ and hence $\tilde{f} \in U_z$.

Problem 9

a) Let $(X, ||\cdot||)$ be a normed space and let x_n be a sequence in X such that $x_n \to 0$ in weak-$\star$ topology. Show that there exists a constant $M > 0$ such thatsup$_n ||x_n|| \leq M$.

b) Show that there exists a normed space $(X, ||\cdot||)$ and a sequence f_n such that $f_n \to 0$ in weak$-\star$ topology, but $||f_n|| \to \infty$.

Solution:

a) Let us identify x_n with the element $\tilde{x_n} \in X^{**}$ such that $\tilde{x_n}(f) = f(x_n)$ for every $f \in X^*$. By uniform boundedness principle, we know that $\sup_n ||x_n|| < \infty$ if $\{f(x_n) : n \in \mathbb{N}\}$ is bounded for every $f \in X^*$.

   To see this, we fix $f \in X^*$ and note that $f(x_n) \to 0$ as $n \to \infty$. In particular, $|f(x_n)| < 1$ for all $n \geq n_0$ for some n_0. Therefore, $\{f(x_n) : n \in \mathbb{N}\}$ is bounded by $\max\{|f(x_1)|, \ldots, |f(x_{n_0})|, 1\}$.

b) Let $X = c_{00}$. We know that $X^* = \ell_1$. Let e_j be the sequence in ℓ_1 with zero in each coordinate expect j-th coordinate and 1 in j-th coordinate. Let $x_n = \sum_{m=n}^{m=2n} e_m$. It is clear that $||x_n||_1 = n$. Now let a_n be a sequence in c_{00}. It is clear that $\sum_{m=1}^{\infty} a_n x_n = \sum_{m=n}^{2n} a_m = 0$ for n large. Therefore, $x_n \to 0$ in weak-$\star$.

Problem 6
If f is analytic in $\mathbb{D}$ and if there are constants $C > 0$ and $0 < \alpha < 1$ such that

$$|f(z) - f(w)| \leq C|z - w|^{\alpha}$$

for all $z, w \in \mathbb{D}$, show that

$$\left|f'(z)\right| \leq C(1 - |z|)^{\alpha - 1}$$

Solution: For $z \in \mathbb{D}$, there exists $\epsilon > 0$, such that $C_\epsilon(z)$-the circle of radius centered at z- lies in $\mathbb{D}$. Then by Cauchy integral formula,

$$f'(z) = \frac{1}{2\pi i}\int_{C_\epsilon(z)} \frac{f(\zeta)}{(\zeta - z)^2}d\zeta = \frac{1}{2\pi i}\int_{C_\epsilon(z)} \frac{f(\zeta) - f(z)}{(\zeta - z)^2}d\zeta + \frac{1}{2\pi i}\int_{C_\epsilon(z)} \frac{f(z)}{(\zeta - z)^2}d\zeta.$$

Note that $\frac{1}{2\pi i}\int_{C_\epsilon(z)} \frac{f(z)}{(\zeta-z)^2}d\zeta = \frac{f(z)}{2\pi i}\int_{C_\epsilon(z)} \frac{d\zeta}{(\zeta-z)^2} = 0$. Thus,

$$|f'(z)| = \left|\frac{1}{2\pi i}\int_{C_\epsilon(z)} \frac{f(\zeta)}{(\zeta - z)^2}d\zeta\right| = \left|\frac{1}{2\pi i}\int_{C_\epsilon(z)} \frac{f(\zeta) - f(z)}{(\zeta - z)^2}d\zeta\right| \leq \frac{1}{2\pi}\int_{C_\epsilon(z)} \left|\frac{f(\zeta) - f(z)}{(\zeta - z)^2}\right||d\zeta|.$$

By the given hypothesis, $\left|\frac{f(\zeta)-f(z)}{(\zeta-z)^2}\right| \leq C|\zeta - z|^{\alpha-2} = C.\epsilon^{\alpha-2}$ on $C_\epsilon(z)$. Thus, it follows that $|f'(z)| \leq \frac{1}{2\pi}C.\epsilon^{\alpha-2}.2\pi.\epsilon = C.\epsilon^{\alpha-1}$. But for $\zeta \in C_\epsilon(z)$, we have $\epsilon = |\zeta - z| \leq (1 - |z|)$, hence $|f'(z)| \leq C(1 - |z|)^{\alpha-1}$, which completes the proof.

Problem 8
Show that there is no analytic function f in the unit disc $\mathbb{D}$ such that $|f(z_n)| \to \infty$ for all sequences $z_n \in \mathbb{D}$ such that $|z_n| \to 1$.

Solution: Assume an analytic function $f : \mathbb{D} \to \mathbb{C}$ exists which satisfies the hypothesis of the question. We first claim f has only finitely many zeroes in $\mathbb{D}$. Suppose not- say there exists a sequence $\{z_k\}_{k\geq1} \subset \mathbb{D}$ such that $f(z_k) = 0$ for all $k \geq 1$. Then since $\overline{\mathbb{D}}$ is compact, this sequence of z_k's has a limit point, say $z_0 \in \overline{\mathbb{D}}$. Since f is continuous and $f(z_k) = 0$ for all k, it follows that $f(z_0) = \lim_{k\to\infty} f(z_k) = 0$. Since $|f(z)| \to \infty$ as $|z| \to 1$, it follows that $z_0 \in \mathbb{D}$. Thus, f is zero on a set of points with a limit point in $\mathbb{D}$, which would imply $f = 0$ on $\mathbb{D}$ by the identity theorem. This is not possible as this violates the hypothesis. Thus, f can have only finitely many zeroes in $\mathbb{D}$- say, $z_1, z_2, \ldots, z_k$, with multiplicities $m_1, m_2, \ldots, m_k$ respectively. Then consider the following function on $\mathbb{D}$:

$$g(z) = \frac{(z - z_1)^{m_1}(z - z_2)^{m_2} \ldots (z - z_k)^{m_k}}{f(z)}$$

Note that clearly, g is holomorphic on $\mathbb{D} \setminus \{z_1, \ldots, z_k\}$. In fact, g is holomorphic on entire $\mathbb{D}$, since in a neighbourhood of z_i, we have $g(z) = (z - z_i)^{m_i}g_i(z)$, where $g_i(z)$ is analytic in that neighbourhood. Furthermore, $|g(z)| \to 0$ as $|z| \to 1$, by the given hypothesis. But then, by the maximum modulus principle, $|g| = 0$ on $\mathbb{D}$, which is contradiction. This implies our desired conclusion.

Problem 1
Evaluate $\int\limits_{\infty}^{\infty} \frac{\cos(xt)}{2+2x+x^2}\, dx$ for $t \in \mathbb{R}$.

Problem 2
Let f_n be a sequence of analytic functions on a domain $G \subseteq \mathbb{C}$. Suppose $f_n \to f$ uniformly on compact subsets of G. Then f is analytic on G, and it is either one-to-one or constant.

Solution: Obviously f must be continuous. Apply Morera's theorem to get that f is analytic.

Let f be non-constant, it suffices to show that if f_n were non-vanishing then f is non-vanishing on G. To do this observe that

$$\int \frac{f_n'}{f_n} \to \int \frac{f'}{f},$$

and $\int \frac{f_n'}{f_n} = 0$ for all n.

Problem 3
The function $f(z) = \sin z - z^2$ has infinitely many zeros in $\mathbb{C}$.

Solution: I particularly like this problem. Before we attempt this problem let us first look at a simpler problem. What if the question was about the zeroes of $f(z) = sin(z) - z$? In this case, one notes that $f(z + 2\pi) = f(z) - 2\pi$. It follows that if f takes the value 0 finitely often then it must take all the values $2k\pi$ only finitely often, which would mean that f is a polynomial. In fact more than this, the fact that $f(z + 2\pi) = f(z) - 2\pi$ implies that f must be surjective (see 1 Problem 5). Finally, it also follows that f assumes all the values infinitely often. complete the solution

Problem 4
Suppose f is entire and $|f(z)| = 1$ for all $z \in \mathbb{R}$. Prove that $f(z) = e^{g(z)}$ for some entire function g.

Solution: The proof is really simple. Recall that it is enough to show that $f(z) \neq 0$ for all $z \in \mathbb{C}$.

To this end, we define $h(z) = \overline{f(\bar{z})}$ (note that h is entire). Now observe that $f(z)h(z) = 1$ on $\mathbb{R}$. By identity principle, it follows that $f(z)h(z) = 1$ on whole $\mathbb{C}$, and therefore, f is non-vanishing.

Problem 5
Suppose f is non-constant entire function such that

$$f(z) = f(z+1) \qquad \text{and} \qquad f(z+i) = e^{4i\pi z} f(z).$$

How many zeroes does f have in the set $R = \{z : 0 \leq \Re(z) < 1, 0 \leq \Im(z) < 1\}$?

Solution: From our hypothesis, we have

$$f'(z+1) = f'(z) \qquad \text{and} \qquad f'(z+i) = -4i\pi f(z) + e^{-4i\pi z} f'(z).$$

Let ∂R be the boundary of rectangle R. Let R_1, R_2, R_3, R_4 be the sides of the rectangle ∂R (starting from $(0,0)$ and going in counter-clockwise direction). Let $n_R(f)$ be the number of zeros of f in R. We know by the Argument principle that

$$(2i\pi)n_R(f) = \int\limits_{\partial R} \frac{f'}{f} dz = \sum_i \int\limits_{R_i} \frac{f'}{f} dz.$$

Now observe that

$$\int\limits_{R_2} \frac{f'}{f} dz + \int\limits_{R_4} \frac{f'}{f} = \int\limits_{R_2} \left(\frac{f'(z)}{f(z)} - \frac{f'(z+1)}{f(z+1)} \right) dz = 0.$$

Also note that,

$$\begin{aligned}\int\limits_{R_1} \frac{f'}{f} dz + \int\limits_{R_3} \frac{f'}{f} &= \int\limits_{R_1} \left(\frac{f'(z)}{f(z)} - \frac{f'(z+i)}{f(z+i)} \right) dz \\ &= \int_{R_1} (4i\pi) dz \\ &= 4i\pi.\end{aligned}$$

It follows that $n_R(f) = 2$.

Problem 6
Let K be a countable closed subset of $\mathbb{C}$. If f is bounded analytic on $\mathbb{C} \setminus K$ then f is constant.

Solution:

Remark 1. *This is an interesting question. Let us look at the following question. If f is a bounded and analytic function on $\mathbb{C} \setminus \{z\}$, then by Riemann's removable singularity theorem we can obtain a (bounded analytic) extension of f on the whole complex plane. By Liouville's theorem, we get that the extension of f is constant and hence so is f. Of course one can immediately extend the result to any bounded analytic function f defined on $\mathbb{C} \setminus K$ where K is a discrete set. The problem goes one step further and claims that the same is true for any closed countable set of $\mathbb{C}$. It is important to note here that a closed countable subset K of C need not be discrete (and it can be pretty nasty.)*

More generally, one can ask the following question. What are all sets K with the property that any bounded analytic function on $\mathbb{C} \setminus K$ is constant? Or equivalently, what are all sets K such that any analytic function on $\mathbb{C} \setminus K$ extends to an analytic function on $\mathbb{C}$? These sets are in some sense 'small sets'. Ahlfors introduced the notion of '(analytic) capacity' of a set, and it turns out that there is a simple and beautiful answer to this question in terms of the analytic capacity of set K. As one might guess, K has capacity 0 if and only if any bounded analytic function on $\mathbb{C} \setminus K$ is constant.

Solution: There are various ways one can proceed to solve this. We give below an amusing solution. Let $\mathcal{F} = \{(g, U) : g \in \mathcal{H}(U), (\mathbb{C} \setminus K) \subseteq U\}$ be the family of bounded holomorphic functions defined on some open set containing $C \setminus K$ which extend f. This is non-empty because we have already given one such function. We now define a partial order on $\mathcal{F}$ by declaring $(f, U) \preceq (g, U')$ if $U \subseteq U'$ and $f = g$ on U. It is clear that any increasing chain in $\mathcal{F}$ has a maximal element. Suppose (f_n, U_n) is increasing chain in $\mathcal{F}$. Set $U = \cap U_n$, and for any $z \in U$ set $f(z) = f_n(z)$ if $z \in U_n$. Clearly, f is well-defined and holomorphic on U. It follows from the Zorn's lemma that $\mathcal{F}$ has a maximal element, say (h, U). Now, the key point is that if U is not $\mathbb{C}$, then there must be an isolated point of $\mathbb{C} \setminus U$ and by Riemann removable singularity theorem h can be extended to that point which contradicts the maximality of (h, U).

The above proof is not particularly elegant because it hides so much.

Another idea is that by Riemann removable singularity theorem, we can extend the function f to all those points which are isolated. This gives us a bounded analytic extension of f_1 to the complement of set $K_1 := K \setminus iso(K)$ where $iso(K)$ is the set of all isolated points of K. But K_1 is a subset of K and hence it is itself countable. We repeat this process, ad-infinitum, to obtain a sequence of function f_n defined on $\mathbb{C} \setminus K_n$ where $K_n = K_{n-1} \setminus iso(K_{n-1})$. We use the construction in the previous solution to define f on $\mathbb{C} \setminus \cap_n K_n$. Since $\cap K_n$ has no isolated points and it is countable, from Baire's category theorem, we know that $\cap K_n = \emptyset$. Therefore, f is defined on whole of $\mathbb{C}$.

Problem 7

If f is analytic on $\mathbb{C} \setminus [0,1] \cup [2,3]$, then $f = g_1 + g_2$ where g_1 is holomorphic on $\mathbb{C} \setminus [0,1]$ and g_2 is analytic on $\mathbb{C} \setminus [2,3]$.

Solution: Let $z \in \mathbb{C} \setminus [0,1]$. Define $g_1(z) = \int\limits_{\gamma_1} \frac{f(w)}{w-z}\, dw$, where γ_1 is any simple closed curve going around $[0,1]$ such that z lies in the unbounded component of $\mathbb{C} \setminus \gamma^*$. This defines (check) a holomorphic function on $\mathbb{C} \setminus [0,1]$. which agrees which f on $\mathbb{C} \setminus [0,1] \cup [2,3]$. Similar define g_2 on $\mathbb{C} \setminus [2,3]$. Note that for any $z \in \mathbb{C} \setminus [0,1] \cup [2,3]$ we have $f = g_1 + g_2$.

Problem 8

For $j = 1, 2$, let f_j be one-one analytic map from $\mathbb{D}$ onto G_j. Assume that $f_j(0) = 0$ for $j = 1, 2$. If $G_1 \subseteq G_2$ then $|f_1'(0)| \leq |f_2'(0)|$.

The conclusion is no longer true if f_j are not injective.

Solution: Define $g : \mathbb{D} \to \mathbb{D}$ by $g(z) = f_2^{-1}(f_1(z))$. It is easy to see that g is holomorphic and $g(0) = 0$. It follows from the Schwarz lemma that $|g'(0)| \leq 1$, which translates to $|f_1'(0)| \leq |f_2'(0)|$.

Take $f_1(z) = z$ and $f_2(z) = z^2$. It is clear that $f_1(\mathbb{D}) = \mathbb{D} = f_2(\mathbb{D})$, but $f_1'(0) = 1$, while $f_2'(0) = 0$.

Problem 1
Evaluate $\int\limits_0^\infty \frac{\cos t}{a^4+t^4}\, dt$ for $a > 0$.

Problem 2
If f has a pole at z_0 then e^f has an essential singularity at z_0.

Solution: By replacing z by $z - z_0$ we can assume that $z_0 = 0$. Note that if 0 is a removable singularity of e^g then 0 must be a removable singularity of g. It follows that 0 can not be a removable singularity of e^f. And 0 can not be a pole of e^f because that would imply 0 is a removable singularity of e^{-f} which in turn would imply that 0 is a removable singularity of $-f$ (and hence f).

Problem 3
Assume that f is meromorphic on a convex open set U of $\mathbb{C}$ with poles at $\{z_j\}_{j\in J}$. Then, there exists meromorphic function g on U satisfying $g'(z) = f(z)$ on $U \setminus \{z_j\}_{j\in J}$ if and only if $\mathrm{Res}(f, z_j) = 0$ for every pole z_j of f.

Solution: Note that the question is essentially about when can define primitive of a meromorphic function. And, it is clear that we can do that if and only if the integrals along two different paths with same end points are equal. The condition that the Residue at each pole is 0 would essentially mean that the integral along any closed contour is 0 (sum of residues 'inside' that contour) and therefore the integral along paths is well defined.

More formally, fix $z_0 \in U \setminus \{z_j\}_{j\in J}$. And for any $z \in U \setminus \{z_j\}_{j\in J}$ define

$$g(z) = \int\limits_{\gamma_z} f(z)dz,$$

where $\gamma_z : [0, 1] \to U \setminus \{z_j\}_{j\in J}$ is any path such that $\gamma_z(0) = z_0$ and $\gamma_z(1) = z$. Note that if γ_z and η_z are two such paths then $\gamma_z - \eta_z$ is a closed contour in U. It follows from tge residue theorem that

$$\int\limits_{\gamma_z} f(z)dz - \int\limits_{\eta_z} f(z)dz = 0.$$

Therefore $g(z)$ is well-defined meromorphic function on U. It is clear from the construction that $g' = f$.

Problem 4
Suppose f and g are holomorphic on an open set containing $\overline{\mathbb{D}}$, and $f(z) \neq 0$, for all $z \in \partial\mathbb{D}$. Assume that

$$\Re\left(\frac{f(z)}{g(z)}\right) \geq 0 \quad \text{for all } z \in \partial\mathbb{D}.$$

Show that f and g have the same number of zeroes in $\mathbb{D}$, counting multiplicities.

Solution: Observe that $h(z) = \log\left(\frac{f}{g}\right)$ is holomorphic and well-defined in a neighborhood of

$|z| = 1$. Therefore, we have

$$0 = \int\limits_{|z|=1} h'(z) = \int\limits_{|z|=1} \left(\frac{f'}{f} - \frac{g'}{g}\right).$$

It follows from the argument principle that f and g have same number of zeroes in the disk.

Alternatively, one can use the Rouche's theorem directly. Note that if $\Re\left(\frac{f(z)}{g(z)}\right) \geq 0$, then we have

$$\begin{aligned}
|\frac{f}{g} - 1|^2 &= \left(|\frac{f}{g}|^2 + 1 - 2\Re\left(\frac{f}{g}\right)\right) \\
&< 1 + |\frac{f}{g}|^2 + 2|\frac{f}{g}| \\
< (1 + \frac{|f|}{|g|}).
\end{aligned}$$

In other words, we get $|f - g| < |f| + |g|$ on the $\partial\mathbb{D}$. It follows from Rouche's theorem that f and g have the same number of zeroes in the $\mathbb{D}$.

Problem 5

Prove from the first principle that

$$f(z) := \sum_{m,n\in\mathbb{Z}} (z + m + ni)^{-3}$$

converges for $z \in \mathbb{C}\setminus\mathbb{Z}[i]$ to a meromorphic function on $\mathbb{C}$. Show that f is $\mathbb{Z}[i]$ periodic and that all the poles of f occur at points in $\mathbb{Z}[i]$ and are all of order 3.

Solution:

Problem 6

Assume f is analytic on a neighborhood of $A = \{z : r < |z| < R\}$, where $0 < r < R < \infty$, and assume that $|f(z)| = 1$ for $z \in \partial A$. Show that if f is not constant then f has at least two zeroes in A (counting multiplicity).

Solution: The key point to show here is that if f is non-constant and $|f(z)| = 1$ on $|z| = x$, then $f(\{|z| = x\})$ winds around the origin at least once. Using this for $|z| = r$ and $|Z| = R$ we see that the winding number of $f(\partial A)$ around the origin is at least 2.

Problem 7

Let G_n be an increasing sequence of simply connected open subsets of $\mathbb{C}$, all containing 0, such that $G = \bigcup\limits_{n=1}^{\infty}$ is not all of $\mathbb{C}$. For each n let f_n be the conformal map of $\mathbb{D}$ onto G_n satisfying $f_n(0) = 0$ and $f_n'(0) > 0$. Prove that:

a) G is simply connected.

b) The sequence $\{f_n\}$ converges normally in $\mathbb{D}$ to a conformal map of $\mathbb{D}$ onto G.

Solution:

a) This is a purely topological fact, but we give an analytical proof. Recall that a domain Ω is simply connected if and only if $\int\limits_\gamma f dz = 0$ for every holomorphic function $f \in H(\Omega)$ and every closed curve γ. Let f be holomorphic in G, and let γ be a closed curve in G. Using the compatness of γ, and the fact that G_n are increasing we obtain n_0 such that γ and its inside are contained in G_n. Since G_n is simply connected, $\int_\gamma f dz = 0$.

b)

Problem 8

1. Suppose $0 < a_k < 1$. Prove that if
$$\prod_1^\infty (1 - a_k) = 0,$$
then $\sum a_k$ diverges.

2. Let $p_k(z)$ be polynomial of degree k with $p_k(0) = 1$ such that p_k has no zeros in $\mathbb{D}(0, k^3)$. Show that $\prod p_k(z)$ converges locally uniformly in $\mathbb{C}$.

Solution:

1. We will prove the contrapositive. Assume that $\sum a_k < \infty$. It follows that $\prod e^{a_k} = e^{\sum a_k} < \infty$, and hence $\prod e^{a_k} > 0$. But for $0 < a_k < 1$ we have $(1 - a_k) > e^{-a_k}$ and hence $\prod(1 - a_k) > e^{-\sum a_k} > 0$.

2. Let K be a compact subset of $\mathbb{C}$. Let $R > 0$ be such that $K \subseteq \mathbb{D}(0, R)$. Let k_0 be large so that $R \leq k_0^3$. For $k \geq k_0$, we know that $p_k(z)$ does not vanish on $\mathbb{D}(0, R)$. Since $p_k(0) = 1$, we write each $p_k(z) = 1 + q_k(z)$ where q_k is a polynomial of degree k with no constant term. We want to show that $\prod_{k \geq k_0}(1 + q_k(z))$ converges uniformly on $\mathbb{D}(0, R)$. In the view of a), it is sufficient to show the uniform convergence of $\sum_{k \geq k_0} q_k(z_0)$ on $\mathbb{D}(0, R)$.

Problem 1
Assume $b > 1$ is a real number. Using contour integrals compute

$$\int_0^\infty \frac{dx}{1+x^b}$$

Express your answer as a positive real number. Justify all estimates.

Solution: Consider $f(z) = \frac{1}{1+z^b}$ as a function of complex variable z. f is holomorphic at points of $\mathbb{C}$, where $1 + z^b$ does not vanish. Now, f has a singularity as points where the denominator vanishes, namely the points $z_k = \exp(i(2k+1)\pi/b)$. Since $|f| \to \infty$, when $z \to z_k$ for any k, it follows that z_k are poles of f. Consider the wedge contour γ_R of radius $R > 1$, given by three parts: (i) $\gamma_{R,1}$, which travels from 0 to R along the real axis, (ii) $\gamma_{R,2}$, which travels from R to $R\exp(i2\pi/b)$ along the arc of circle $C_R(0)$ of radius R centered at 0, and finally, (iii) $\gamma_{R,3}$, which travels from $R\exp(i2\pi/b)$ to 0 inwards along the radius of the circle $C_R(0)$, joining these two points. Then the region bounded by the closed contour γ_R contains exactly one pole of f, namely, $z_1 = \exp(i\pi/b)$. Thus, by the residue formula,

$$\int_{\gamma_R} f(z)dz = \int_{\gamma_{R,1}} f(z)dz + \int_{\gamma_{R,2}} f(z)dz + \int_{\gamma_{R,3}} f(z)dz = 2\pi i Res(f; z = z_1)$$

Now note the following, $(b > 1)$

$$\left| \int_{\gamma_{R,2}} f(z)dz \right| = \left| \int_0^{2\pi/b} \frac{iR\exp(i\theta)}{1 + R^b \exp(ib\theta)} d\theta \right| \leq \int_0^{2\pi/b} \frac{R}{1+R^b} |d\theta| = \frac{2\pi R}{1+R^b} \to 0 \text{ as } R \to \infty,$$

We also have, $\int_{\gamma_{R,1}} f(z)dz = \int_0^R \frac{1}{1+x^b} dx$ and $\int_{\gamma_{R,2}} f(z)dz = \int_R^0 \frac{\exp(i2\pi/b)}{1+x^b} dx$, since $\gamma_{R,3}$ is parametrized by $\gamma_{R,3}(t) = \exp(i2\pi/b)t$ from $t = R$ to $t = 0$. Now for residue calculation, note that $\lim_{z\to z_1}(z - z_1)f(z) = \lim_{z\to z_1} \frac{z-z_1}{z^b - z_1^b} = \frac{1}{bz_1^{b-1}}$ Thus, we have,

$$(1 - \exp(i2\pi/b)) \int_0^R \frac{1}{1+x^b} dx + \int_{\gamma_{R,2}} f(z)dz = 2\pi i Res(f; z = z_1) = \frac{2\pi i}{bz_1^{b-1}} = \frac{-2\pi i \exp(i\pi/b)}{b}$$

Taking limit $R \to \infty$, from the above observations, we get:

$$\int_0^\infty \frac{1}{1+x^b} dx = \frac{2\pi i}{b(\exp(i\pi/b) - \exp(-i\pi/b))} = \frac{\pi}{b\sin(\pi/b)}$$

Problem 2
Suppose f is a non constant entire function such that $f(1 - z) = 1 - f(z)$ for every $z \in \mathbb{C}$. Show that f assumes every complex number.

Solution: Recall from the (little) Picard's theorem that a non-constant entire function can not miss two values. The idea here is to show that if f misses one complex value then it must miss another. More precisely, observe that if f misses $\alpha \in \mathbb{C}$, then f must also miss $1 - \alpha$. (Because if $f(z_0) = 1 - \alpha$ then $f(1 - z_0) = 1 - (1 - f(z_0)) = \alpha$). Therefore, we will be done if $\alpha \neq 1 - \alpha$. But note that $\alpha = 1 - \alpha$ if and only if $\alpha = \frac{1}{2}$. But $f(0) = \frac{1}{2}$.

Problem 3
Let f and g be analytic functions on bounded domain $G \subseteq \mathbb{C}$, and continuous on $\overline{G}$. Show that $|f(z)| + |g(z)|$ achieves its maximum on the boundary.

Solution: This is an interesting problem which generalizes the maximum modulus principle. Note that by MMP we can say that $|f(z) + g(z)|$ achieves its maximum on the boundary, but this is not quite what the problem asks for.

Since $|f(z)| + |g(z)|$ is continuous, it must achieve its maximum on $\overline{G}$. Let $\max_z(|f(z)| + |g(z)|) = |f(z_0)| + |g(z_0)|$. Fix a $\lambda \in S^1$ such that $|f(z_0)| + |g(z_0)| = |\lambda f(z_0) + g(z_0)|$. Now observe that by MMP applied to $\lambda f + g$ we know tha the maximum of $|\lambda f(z) + g(z)|$ is achieved on the boundary. But $|\lambda f(z_0) + g(z_0)| \le \max |\lambda f(z) + g(z)| \le \max(|f(z)| + |g(z)|) = |f(z_0)| + |g(z_0)| = |\lambda f(z_0) + g(z_0)|$. It follows that z_0 must be a boundary point.

Remark 2. *It can be easily seen that the proof above can be modified (or used inductively) to argue that if $f_i, i \in \{1, \dots, n\}$ are holomorphic functions on G such that each f_i is continuous on $\overline{G}$, then $|f_1(z)| + \ldots + |f_n(z)|$ achieves its maximum on the boundary.*

Problem 4

a) Find a bounded harmonic function u that is continuous on $S = \{z : |z| \le 1, \mathfrak{Im}(z) \ge 0, z \ne \pm 1\}$ such that $u = 3$ on the interval $(-1, 1)$ of real axis and $u = 1$ on $\{z : |z| = 1, \mathfrak{Im}(z) > 0\}$.

b) There are infinitely many *unbounded* harmonic functions with the properties stated above. Find two of them.

Solution:

a

b

Problem 5
Is there an analytic function $f : \mathbb{D} \to \mathbb{D}$ such that $f(\frac{1}{2}) = \frac{2}{3}$ and $f(\frac{1}{4}) = \frac{1}{3}$.

Solution: Use Schwarz-Pick lemma to reach a contradiction.

Problem 6
Prove that there is no $1 - 1$ holomorphic function f from the punctured disk $\mathbb{D} \setminus \{0\}$ onto the annulus $A = \{z : 0 < a < |z| < b\}$.

Solution: Any such f would have a removable singularity at 0 and hence (by open mapping theorem) would extend to a holomoprhic map $f : \mathbb{D} \to A$. Now let $w = f(0) \in A$. Since f was already an onto map from punctured disk onto A, there exist a $0 \ne z \in \mathbb{D}$ such that $f(z) = f(0) = w$. Choose disjoint open balls around 0 and z, say B_0 and B_z. Note that $f(B_0) \cap f(B_z)$ is a non-empty open subset of A (it contains w and hence non-empty). This contradicts the fact that f was one-to-one on $\mathbb{D} \setminus \{0\}$.

Alternatively, by open mapping theorem any such function f would give a bi-conformal map between $\mathbb{D} \setminus \{0\}$ and A. One can use the fact that $A(r_1, r_2)$ and $A(r_1', r_2')$ are conformally equivalent if and only if $\frac{r_1}{r_2} = \frac{r_1'}{r_2'}$.

Problem 7
Suppose f and g are non-zero entire functions such that $|f(z)| \leq |g(z)|$ when $|z| \geq 2017$. Prove that $\frac{f}{g}$ is a rational function.

Solution: Let $a_j, i = 1, \ldots, k$ be the zeroes of g in $\overline{\mathbb{D}(0, 2017)}$. Let $P(z) = \prod_{j=1}^{k} (z - a_j)$. Let $h(z)$ be the entire function such that $g(z) = P(z)h(z)$ (note that $h(z)$ does not have any zero in $\overline{\mathbb{D}(0, 2017)}$). It follows that $\frac{f}{h}$ such that $|\frac{f}{h}(z)| \leq |P(z)|$ for all $|z| \geq 2017$. It follows therefore that $\frac{f}{h}$ is a polynomial (use Cauchy's estimate to show that all the derivative of order $\geq k+1$ vanish), say $Q(z)$. Hence, $\frac{f}{g} = \frac{Q}{P}$ which is a rational function.

Problem 8
Let $\Omega = \{z : |z| < 1\} \cup \{z : |z - 1| < 1\}$. Prove that there is a function f that is analytic on Ω but does not extend to any *open set* that contains Ω as a proper subset.

Solution: As written the question is wrong. If we take $\Omega' = \{z : |z - 2017| < 1\}$. Then any analytic function f on Ω of course can be extended (by setting it equal to any arbitrary constant on Ω') to $\Omega \cap \Omega'$ which contains Ω properly. However, the it is indeed true if *open set* in the question is replaced by a *domain* (recall that a domain is an open connected subset of $\mathbb{C}$).

To see this choose a countable dense subset $\{a_j\}_{j \in \mathbb{N}}$ of $\partial\Omega$. And, let f be the Weirestrass function such that $f(a_j) = 0$ for all $j \in \mathbb{N}$.

Problem 1
Evaluate
$$\int_0^\infty \frac{\sqrt{x}\log x}{(1+x)^2}dx.$$

Problem 2
Find all function f analytic in the interior of the the rectangle with vertices $1+i, 1-i/2, -1/2-i/2, -1/2+i$ such that $f(z+1)=f(z)$ and $f(z+i)=1+f(z)$ wherever these quantities are defined.

Proof. First of all observe that f can be extended continuously on the boundary of the rectangle, and consequently it can be extended on the entire plane. Morera's theorem then shows that the f is entire. Therefore, the question reduces to finding all entire functions f such that $f(z+1)=f(z)$ and $f(z+i)=1+f(z)$. Now observe that $f'(z+1)=f'(z)=f'(z+i)$. Therefore, $f'(z)$ is bounded, and hence constant by Liouville's theorem. Consequently, $f(z)=az+b$ for some constants $a,b\in\mathbb{C}$.

Now, note that $f(z+1)=a(z+1)+b=az+b+a=f(z)+a$. Which would mean $a=0$. Which is impossible, because any constant function can not satisfy $f(z+i)=1+f(z)$. It follows that there are no such functions. □

Problem 3

Problem 4

Problem 5

Problem 6

Problem 7
Suppose f is a function defined on the unit disk $\mathbb{D}$ with the property that for each triple of points $a,b,c\in\mathbb{D}$ there is analytic function g which is bounded by 1 and satisfies $f(a)=g(a), g(b)=f(b)$ and $g(c)=f(c)$. Prove that f is analytic in $\mathbb{D}$ and is bounded by 1.

Solution: The fact that f is bounded is trivial, therefore, we only show that f indeed is analytic.

To this end, we fix $z_0\in\mathbb{D}$ and we will show that f is differentiable at z_0. Let us write the dfifference quotient
$$D_h(f)=\frac{f(z_0+h)-f(z_0)}{h}.$$

The idea here is that if we take any sequence $h_n \to 0$, then we can find a sequence of holomorphic functions g_n such that $g_n(z_0 + h_n) = f(z_0 + h_n)$ and $g(z_0) = f(z_0)$, which gives $D_{h_n}(f) = D_{h_n}(g_n)$. Using the fact that the family g_n is uniformly bounded by 1, and hence normal, we can extract a subsequence h_{n_k} such that $g_{n_k} \to h$ uniformly on compact subsets for some holomorphic function h on $\mathbb{D}$. The uniform convergence would of course yield that $\lim_{k\to\infty} D_{h_{n_k}}(f)$ exists and is finite. The issue, however, is that if h_n and $\tilde{h}_n$ are two different sequences, then potentially we may get two different limits $D_{h_{n_k}}(f)$ and $D_{\tilde{h}_{n_j}}(f)$. But note that we have only used the part of the hypothesis, all we used is that g_n agrees with f at two points. We now use the full hypothesis two show that given two sequences $h_n \to 0$ and $k_n \to 0$, we can obtain a subsequence h_{n_i} and k_{n_j} such that the limits $D_{h_{n_i}}(f)$ and $D_{k_{n_j}}(f)$ both exist and are equal. It would follow therefore that the limit $D_{h_n}(f)$ exists for any sequence $h_n \to 0$, because we already know that the limit $D_{h_{n_j}}$ exists along some subsequence, say, $D_{h_{n_j}}(f) \to a$. If the limit $D_{h_n}(f)$ does not exist, then we can find a subsequence, which we will call k_n, such that $|D_{k_n} - a| > \epsilon$ for some $\epsilon > 0$. Now, using h_{n_j} and k_n as the original two sequences, we get a contradiction.

Let h_n and k_n be two sequences going to 0. Let g_n be holomorphic function bounded by 1 such that $g_n(z_0 + h_n) = f(z_0 + h_n), g_n(z_0 + k_n) = f(z_0 + h_n)$ and $g(z_0) = f(z_0)$. As argued above, g_n is normal and hence has a convergent subsequence. Let n_k be a subsequence such that $g_{n_k} \to \tilde{g}$ for some holomorphic function g bounded by 1. From the uniform convergence over compact subsets we obtain

$$D_{h_{n_j}}(f) = D_{h_{n_j}}(g_n) = \tilde{g}'(z_0) = D_{k_{n_j}}(g_n) = D_{k_{n_j}}(f).$$

Problem 8

Problem 1
Evaluate the following integrals

$$\int_0^\infty \frac{\log x}{1+x^2}\,dx \quad \text{and} \quad \int_0^\infty \frac{(\log x)^2}{1+x^2}\,dx.$$

Problem S
tate the definition of local uniform convergence of infinite product $\prod_{n=0}^\infty f_n(z)$ of holomorphic functions on a domain G. Let $p(z) = z^z + z + 1$ and show that

$$P(z) = \prod_{n=0}^{\infty} p\left(z^{3^n}\right)$$

converges locally uniformally on $\mathbb{D}$. Also, prove that $P(z) = \frac{1}{1-z}$ for all $|z| < 1$.

Problem 3
How many roots does $p(z) = 2z^7 - 4z^3 + 1 = 0$ have in annulus $A = \{1 < |z| < 2\}$.

Solution: It is easy to see that the $p(z)$ does not have a zero on ∂A. Apply Rouche's theorem to obtain that $p(z)$ have same number of zeroes in A as does $q(z) = 2z^7 - 4z^3$. And, it is easy to see that there are 4 zeroes of q in A.

Problem 4
Suppose $G \subseteq \mathbb{C}$ is a domain, and $f_n : G \to \mathbb{C}$ is a sequence of holomorphic functions satisfying

$$\int\int_G |f_n(x+iy)|^2 \, dx \, dy = 1$$

for each n. Then, there is a subsequence of f_n which converges to a holomorphic function.

Solution: Note that the problem essentially says that $\mathcal{F} = \{f \in \mathcal{H}(\Omega) : \int\int_G |f_n(x+iy)|^2 dxdy = 1\}$ is a Normal family. This should remind you of Montel. Recall from Monetl's theorem that is suffices (actually it is equivalent) to show that $\mathcal{F}$ is locally uniformly bounded.

To show that $\mathcal{F}$ is locally uniformly bounded. Let $z \in G$, and let R be such that $D(z, 2R) \subseteq G$. (Hence if $w \in D(z,R)$, then $D(w,R) \subseteq G$.) Use the Cauchy's formula to obtain for any $w \in D(z,R)$

$$\begin{aligned} |f(w)| &\leq \frac{1}{\pi R^2} \int \int_{D(w,R)} |f(x+iy)| dxdy \\ &\leq \frac{1}{\sqrt{\pi} R} \left(\int\int_G |f(x+iy)|^2 dxdy \right)^{1/2} = \frac{1}{\sqrt{\pi} R}. \end{aligned}$$

(The second inequality follows from Cauchy-Schwarz).

Problem 5
Suppose f is a nowhere vanishing holomorphic function on the punctured disk $\mathbb{D} \setminus \{0\}$. Prove that there is an integer n and a holomorphic function g on the punctured disk such that $f(z) = z^n e^{g(z)}$.

Solution: This is an interesting problem. To motivate our solution let us first ask ourselves if $f(z) = z^n e^{g(z)}$ can we recover n and g? To give a comparison, recall that a non-vanishing function on f on $\mathbb{C}$ can be written as e^g. And, in that case we recover g by taking a primitive of $\frac{f'}{f}$. The proof essentially boils down to showing that the primitive of $\frac{f'}{f}$ can be defined.

Now coming back to our problem assume that $f(z) = z^n e^{g(z)}$. This gives us that $\frac{f'}{f} = \frac{n}{z} + g'(z)$. It is immediate from this that $n = \frac{1}{2i\pi} \int\limits_{C_r} \frac{f'}{f} dz$. And once we have got n, we can obtain g by integrating $g'(z) = \frac{f'}{f} - \frac{n}{z}$. All we need to do now is to justify these steps.

Problem 6
Consider the domain $G = \{z \in \mathbb{C} : \Im(z) > 0 \text{ and } |z - i| > 1/2\}$. Determine all harmonic functions $u : G \to \mathbb{R}$ that are bounded and harmonic on G and satisfy $u(x) = 1$ for all $x \in \mathbb{R}$ and $u(z) = 2$ for $\{|z - i| = 1/2\}$.

Problem F
ind a conformal map from $\mathbb{D}$ onto $G = \{z \in \mathbb{D} : \Im(z) > 0, |z - 1/2| > 1/2\}$.

Problem 8
Suppose f is holomorphic and bounded in $\mathbb{C} \setminus \mathbb{D}$. If f is real valued on the vertical line segment $\{-2 + iy : 0 \le y \le 1\}$, then f is a constant.

Solution: This is an easy consequence of Schwarz's reflection principle. Look at the restriction of f on the region $\{z + iy : z \le 2\}$, denoted by f_1. By Schwarz reflection principle, we have a holomorphic extension of f_1 on the whole complex plane, which can be written as

$$F(z) = \begin{cases} f(z), & \mathfrak{Re}(z) \le -2 \\ \overline{f(-\bar{z} - 4)}, & \mathfrak{Re}(z) \ge -2 \end{cases}.$$

By identity principle F must agree with f on $\mathbb{C} \setminus \{D\}$. Since f was bounded, it is clear from the formula for F that F is bounded and hence constant (by Liouville's theorem).

Problem 1
Calculate
$$\int_0^\infty \frac{\cos x - 1}{x^2} dx.$$

Problem 2
Suppose f is analytic on $\mathbb{D}$ and continuous on $\overline{\mathbb{D}}$. If $f(z) = 0$ on an arc of the circle $|z| = 1$, then $f \equiv 0$.

Solution: Without loss of generality assume that the arc on which f is zero contains the arc $\{z \in S^1 : 0 \leq \arg(z) \leq \frac{\pi}{N}\}$ for some N. Now define a new function g on the disk by $g(z) = \prod_{k=0}^{N-1} f(e^{2i\pi k/n} z)$. Clearly g is analytic on disk and continuous on the closure of the disk. Note that $g \equiv 0$ on the unit circle. Therefore by the maximum modulus principle $g \equiv 0$ on the disk. It follows that f must be identically zero.

Problem 3
Let $\mathcal{F}$ be the class of holomorphic functions f in the slit half-plane
$$S = \{z \in \mathbb{C} : \Re(z) > 0\} \setminus (0, 1]$$
satisfying $|f(z)| < 1$ for all $z \in S$ and $f(\sqrt{2}) = 0$. Find $f \in \mathcal{F}$ such that $|f(4)| = \sup_{g \in \mathcal{F}} |g(4)|$.

Problem 4
Suppose a_n is a sequence of distinct complex numbers with no limit point, and b_n is an arbitrary sequence of complex numbers. Prove tha there is an entire function f with $f(a_n) = b_n$, for $n = 1, 2, 3, \ldots$.

Problem 5
Suppose f is analytic on $\mathbb{D}$ except for finitely many poles. If $\lim_{z \to e^{i\theta}} |f(z)| = 1$ for every $\theta \in [0, 2\pi)$ then f is a rational function.

Solution: It is a standard application of Blashcke's product. Let α_k be the zeroes of f in $\mathbb{D}$ and let β_j be the poles of f. Define the function
$$B(z) = \prod_k \frac{z - \alpha_k}{1 - \bar{\alpha}_k z} \quad C(z) = \prod_j \frac{z - \beta_j}{1 - \bar{\beta}_k z}.$$
Define $g = \frac{C(z)}{B(z)} f(z)$. Then, it follows from Removable singularity theorem that g is analytic and non-vanishing on the disk. From the property of the Blashke's product, we also know that $|g(z)| = 1$ on the unit circle. By Maximum modulus and minimum modulus principle, it follows that g is a constant. Hence, there exists $\lambda \in S^1$ such that $g = \lambda$. Therefore, $f = \lambda \frac{B(z)}{C(z)}$.

Problem 6
Suppose f_n is a sequence of holomorphic functions in $\mathbb{D}$ for which $u(z) = \lim \Re(f_n(z))$ exists uniformly on every compact subset of $\mathbb{D}$. Assume that there is a point z_0 such that $v(z_0) = \lim Im(f_n(z_0))$ exists. Then, f_n converges uniformly on every compact subset of $\mathbb{D}$.

Problem 7
Suppose $\phi : [-1, 1] \to \mathbb{C}$ is a continuous function, and define $f : \mathbb{C} \setminus [-, 1] \to \mathbb{C}$ by

$$f(z) = \int_{-1}^{1} \frac{\phi(t)}{t - z} dt.$$

Then, f is holomorphic on $\mathbb{C} \setminus [-1, 1]$. Find the Laurent series of f about infinity in terms of ϕ.

Problem 8
For any positive integer n, define $f_n : \mathbb{C} \setminus \{0\} \to \mathbb{C}$ by

$$f_n(z) = 1 + \frac{1}{z} + \frac{1}{2!z^2} + \ldots + \frac{1}{n!z^n}.$$

Let $R > 0$ be given. For sufficiently large n, all the zeros of f_n lie inside the disk $|z| < R$.

Proof. Note that $f_n \to e^{1/z}$ uniformly on compact subsets of $\mathbb{C} \setminus \{0\}$. Since $e^{1/z}$ has removable singularity at ∞. The zeroes of f_n all must converge to 0.

Alternatively, the same argument can be presented as follows. Consider the polynomial $p_n(z) = f_n(1/z)$ defined on $\mathbb{C} \setminus \{0\}$. Then $p_n \to e^z$ unfiromly on compact subsets of $\mathbb{C}$. It follows that given $R > 0$, for sufficiently large all the zeroes of $f_m, m \geq n$ are outside the disk of radius $\frac{1}{R}$. It follows that all zeroes of $f_m, m \geq n$ lie inside the disk of radius R.

Finally, one can use Nivenlinna theory to get a precise bound on the location of zeroes. $\square$

Problem 1

Is there a conformal map from the slit strip

$$D = \{x + iy : -1 < y < 1\} \setminus \{x + iy : x \leq 0, y = 0\}$$

onto $\mathbb{D}$? If yes, find one. If no, prove there is none.

Problem 3
Let f be analytic in the closed unit disc, with $f(-\log 2) = 0$ and

$$|f(z)| \leq e^z$$

for all z with $|z| = 1$. How large can $|f(\log 2)|$ be? Find the best possible bound upper bound.

Solution: For notational simplicity, I will denote $-\log 2$ by α. Let Φ be the Möbius transform defined by taking 0 to α, that is, $\Phi(z) = \frac{\alpha - z}{1 - \alpha z}$. Note that the function $F = e^{-1} f \circ \Phi$ is a holomorphic function taking disk into disk and $F(0) = 0$. By Schwarz lemma we have $|F(z)| \leq |z|$. In particular, we get $|f(\log 2)| \leq e|\Phi^{-1}(\log 2)| = \frac{2e \log 2}{1 + (\log 2)^2}$.

This is not the best solution. Bound is probably never achieved. Because Schwarz lemma says that the the upper bound will be achieved only when $f(z) = e\Phi(z)$. But the hypothesis in the function does not allow this.

Alternatively, we observe that e^z is non-vanishing and hence $G = \frac{f \circ \Phi(z)}{e^z}$ is a holomorphic function that maps disk into disk and sends 0 to 0. We can apply Schwarz lemma on G to obtain $|G(z) \leq |z|$. In particular, $|f(\log 2)| \leq |e^{\Phi(\log 2)}||\Phi(\log 2)|$. This bound is actually achieved precisely when $f(z) = \Phi(z)e^{\Phi(z)}$.

Problem 4
How many roots of the equation $z^4 + 8z^3 + 3z^2 + 8z + 3 = 0$ lies in the right half plane?

Problem 5
Let $\mathbb{N} = \{1, 2, 3 \ldots, \}$. Find an explicit series representation for a meromorphic function on $\mathbb{C}$, which is holomorphic on $\mathbb{C} \setminus \mathbb{N}$, and which has at each $n \in \mathbb{N}$ a simple pole with residue n.

Problem 6

Problem 7
See Problem 4 in 1.

Problem 2
If $a > 1$, prove that $z + e^{-z}$ assumes the value a exactly once in $\mathbb{H} = \{z : \Re(z) > 0\}$.

Problem 3
Let f be an entire function with the property that for every z there is n such that $f^{(n)}(z) = 0$. Show that f is a polynomial.

Solution: For each n let $A_n = \{z : f^{(n)}(z) = 0\}$. Since $\cup A_n = \mathbb{C}$ it follows that there is an n for which A_n is uncountable and hence it has a limit point. It follows by the identity principle that $f^{(n)} \equiv 0$. Therefore, f is a polynomial.

Problem 8
Let D be a bounded domain and $f : D \to D$ be analytic. If f has fixed point $f(z_0) = z_0$ such that $|f'(z_0)| < 1$, then the sequence of iterates $f_n := f \circ f \circ \cdots \circ f$ converges uniformly on compact subsets of D.

Problem 2
Suppose f is a non-constant entire function satisfying $|f(z)| \leq e^{\sqrt{|z|}}$ for all $z \in \mathbb{C}$. For $R > 0$, ;et $n(R)$ be the number of zeroes of f with modulus less than or equal to R. Show that ther exists non-negative constants A, B such that $n(R) \leq A + B\sqrt{R}$.

Problem 3
Let $\mathcal{H}$ be the class of all analytic functions on disk such that $f(0) = 0, f'(0) = 1$ and $|f(z)| \leq 100$ for all $z \in \mathbb{D}$. Prove that there exists a constant $r > 0$ such that $D_r \subseteq f(\mathbb{D})$ for all $f \in \mathcal{H}$.

Solution: Note that by open mapping theorem we already know that for a given f we can find $r > 0$ such that $\mathbb{D}_r \subseteq f(\mathbb{D})$. The thrust of the question lies in the fact that r can be chosen uniformly over $\mathcal{H}$. To get any such result, the first thing to do would be to estimate the value of r for a given function f. The idea is to obtain a lower bound on $\inf\limits_{z\in\mathbb{D}} |f(z)|$. Note that if we could prove that for any $f \in \mathcal{H}$, we have $|f(z)| \geq \alpha > 0$ whenever $|z| = r$. It will follows that $f(\mathbb{D})$ contains $\mathbb{D}(0, \alpha)$.

Let $f(z) = z + \sum\limits_{n=2}^{\infty} a_n z^n \in \mathcal{H}$. By Cauchy's estimate, we obtain $|a_n| \leq 100$. For $|z| = r$, we have

$$|f(z)| \geq r - 100 \sum_{n=2}^{\infty} r^n = r - 100 \frac{r^2}{1-r}.$$

In particular, taking $r = 1/100$, we obtain that $|f(z)| \geq 88/990$. It follows that $\mathbb{D}(0, 88/990) \subseteq f(\mathbb{D}(0, 1/100)) \subseteq f(\mathbb{D})$.

It is worth mentioning that we are losing constants at various places in the above proof. Assume $|f(z)| \leq M$ on $\mathbb{D}$. One can use a more refined version of Cauchy's estimate, that is, $|a_n| \leq \frac{M}{R^n}$ for every $0 < R < 1$. Now fixing $0 < r < R < 1$, and using the same trick as above one can argue that on $|z| = r$, we have

$$\begin{aligned} |f(z)| &> r - M \sum_{n=2}^{\infty} \left(\frac{r}{R}\right)^n \\ &= r - \frac{Mr^2}{R(R-r)}. \end{aligned}$$

One can now choose r and R so as to maximize the last expression. Choosing $R = 1, r = (4M)^{-1}$ one can get $|f(z)| \geq \frac{1}{6M}$. This is probably not the best. But I have not checked.

Problem 4
Suppose f is analytic and bounded on a bounded region $u \subseteq \mathbb{C}$, and continuous on the closure of U. Suppose $|f(z)| = 1$ whenver $z \in \partial U$. Prove that either f is a constant, or $f(U) = \mathbb{D}$.

Solution: First of all note that $f(U) \subseteq \mathbb{D}$ by MMP. Let $w \in \mathbb{D}$ such that $f(z) \neq w$. Then $g(z) = \frac{1}{f(z)-w}$ is a holomorphic function U. Apply MMP to obtain that $|f(z) - w| \geq 1 - |w|$. It follows that $\mathbb{D} \setminus f(U)$ is open (hence $f(U)$ is closed in $\mathbb{D}$). By open mapping theorem $f(U)$ is open. Since $\mathbb{D}$ is connected, it follows that $f(U) = \mathbb{D}$.

Note that one also gets that $f(\overline{U}) = \overline{\mathbb{D}}$.

Problem 5
See 1 Problem 2.

Problem 6

Problem 1

$$\int_0^\infty \frac{\cos(ax)}{(1+x^2)^2}\,dx \qquad \text{for } a>0.$$

Problem 2
Find all conformal maps from $\mathbb{D}$ onto the region $U = \{z : |z| < 1 \text{ and } \left|z - \frac{1}{2}\right| > \frac{1}{2}\}$.

Problem 3
Let $p(z) = az^4 + bz + 1$ for $a, b \in \mathbb{R}$. Find the maximum number of roots of p in the annulus $A = \{z : 1 < |z| < 2\}$, provided $a \in [1, \pi]$ and $b \in [2\pi - 2, 7]$.

Solution: Apply Rouche's theorem with $p(z)$ and $f(z) = az^4$ to conclude that p has 4 zeroes in $\mathbb{D}(0,2)$. Apply Rouche's theorem with $p(z)$ and $f(z) = bz + 1$ to obtain that the numer of zeroes of p in $\mathbb{D}(0,1)$ is 1. Hence, the number of zeroes of p in A is 3.

Problem 4
See Problem 6 in 1.

Problem 5

a State and probe Schwarz Lemma.

b Given $f : \mathbb{D} \to \mathbb{D}$ an anlytic function, show that

$$\frac{|f'(z)|}{1-|f(z)|^2} \le \frac{1}{1-|z|^2}, \qquad \forall z \in \mathbb{D}.$$

Solution: Standard.

Problem 6
Let M_n be a sequence of positive numbers and define a family of functions

$$\mathcal{F} := \{f : f \text{ is analytic on } \mathbb{D} \text{ and } |f^{(n)}(0)| \le M_n\}.$$

Show that $\mathcal{F}$ is normal if and only if $\sum_{n=0}^{\infty} \frac{M_n}{n!} z^n$ converges on $\mathbb{D}$.

Solution: Assume that $\sum_{n=0}^{\infty} \frac{M_n}{n!} z^n$ converges on $\mathbb{D}$ and let $\{f_n\} \subseteq \mathcal{F}$. After passing to a diagonal sequence, we may assume that $f_n^{(k)}(0)$ converges for each k, Let $f_n^{(k)}(0) \to b_k$ for each k. It is clear that $|b_k| \leq M_k$. By Weierstrass's M-test we conclude that $f(z) := \sum\limits_{k=0}^{\infty} \frac{b_k}{k!} z^k$ is a holomorphic function. Let $K \subseteq \mathbb{D}$ be a compact subset. It is easily seen that

$$|f_n(z) - f(z)| \leq \sum_{k=0}^{\infty} |b_k - f_n^{(k)}(0)| \frac{z^k}{k!}.$$

It follows that f_n converges to f uniformly on compact subsets of $\mathbb{D}$.

Conversely is trivial. Just take $h_i = \frac{M_i}{i!} z^i$. Let $f_n = \sum\limits_{i \leq n} h_i(z)$. It is clear that $f_n \in \mathcal{F}$ for each n. But no subsequence of f_n can be convergent.

Problem 7
Show that $\sin(z) - z^2$ has infinitely many zeroes in $\mathbb{C}$.

Solution: see 1 Problem 3.

Problem 8
Given a subharmonic function $u : \mathbb{C} \to \mathbb{R}$, recall that

$$u(z) \leq \frac{1}{2\pi} \int_0^{2\pi} u(z + re^{i\theta})\, d\theta,$$

for any z and for any $r > 0$. Let u be subharmonic, and let $M(r) = \max\limits_{|z|=1} u(z)$.

a) Prove that for any $0 < r_1 \leq |z| \leq r_2$,

$$u(z) \leq \frac{\log r_2 - \log |z|}{\log r_2 - \log r_1} M(r_1) + \frac{\log |z| - \log r_1}{\log r_2 - \log r_1} M(r_2).$$

b) Show that $\lim\limits_{r \to \infty} \frac{M(r)}{\log(r)}$ exists (possibly infinite).

Solution: Standard Harnack's inequality proof.

Problem 1
Let $p(z)$ be the polynomial given by $p(z) = z^n + c_{n-1}z^{n-1} + \ldots + c_1 z + c_0$. Then all the zeros of p lies in a disk centered at 0 with radius

$$R = \sqrt{1 + |c_{n-1}|^2 + \ldots + |c_0|^2}.$$

Solution: Note that if $R = 1$ then 0 is the only root of polynomial p, andthere is nothing to prove.

In the following we assume $R > 1$. From tringle inequality we get $|p(z)| \geq |z|^n - |c_{n-1}z^{n-1} + \ldots + c_1 z + c_0|$. Therefore suffices to show that if $|z| \geq R$ then $|c_{n-1}z^{n-1} + \ldots + c_1 z + c_0| < |z|^n$. This follows from Cauchy-Schwarz inequality as follows

$$\begin{aligned}
|c_{n-1}w^{n-1} + \ldots + c_0|^2 &\leq (c_{n-1}^2 + \ldots + |c_0|^2)(|w|^{2(n-1)} + \ldots + |w|^2 + 1) \\
&(c_{n-1}^2 + \ldots + |c_0|^2)\left(\frac{|w|^{2n-1} - 1}{|w|^2 - 1}\right) \\
&= \frac{R^2 - 1}{|w|^2 - 1}(|w|^{2n-1} - 1) \\
&\leq |w|^{2n-1} - 1 < |w|^{2n}.
\end{aligned}$$

Problem 2
Prove that there exists a sequence of polynomials p_k such that

$$\lim_{k\to\infty} p_k(z) = \operatorname{sgn} z := \begin{cases} 1, & \text{if } \Re(z) > 0 \\ 0, & \text{if } \Re(z) = 0 \\ -1, & \text{if } \Re(z) < 0 \end{cases}.$$

Solution: It is a tricky application of Runge's theorem. Let K_n be the compact set defined as

$$K_n = \left(\{z : \Im(z) = 0\} \cup \{z : |\Im(z)| \geq \frac{1}{n}\}\right) \cap \overline{\mathbb{D}}(0, n).$$

Observe that $\mathbb{C} \setminus [K_n]$ is connected and that $\operatorname{sgn} z$ is a holomorphic function in a neighborhood of K_n. By Runge's theorem there is a sequence of polynomials q_m converging uniformly to $\operatorname{sgn} z$ on K_n. Choose m large so that $|q_m(z) - \operatorname{sgn} z| \leq \frac{1}{n}$ on K_n, and set $p_n = q_m$. It is clear that the sequence of polynomials p_m obtained this way converge to $\operatorname{sgn} z$ (the convergence is no more uniform) on $\bigcup_n K_n = \mathbb{C}$.

Problem 3
If f is analytic map of $\mathbb{D}$ into $\mathbb{D}$ with $f(0) = 0$ and $f'(0) = \frac{1}{2}$, then $f(\mathbb{D})$ contains a disk centred at 0 with radius $7 - 4\sqrt{3}$.

Solution: It is very similar to Problem 3 in 1. Let $f(z) = \frac{z}{2} + \sum_{n\geq 2} a_n z^n$. From Cauchy's estimate, we know that $|a_n| < 1$ for all $n \geq 2$. We now observe that for $|z| = r$, we have

$$|f(z)| \geq \frac{r}{2} - \frac{r^2}{1-r} =: M(r).$$

We now try to find r which will maximize $M(r)$. To this end, we note that $M'(r) = 0$ yields $(1-r)^2 - 2r^2 - 4r(1-r) = 0 = 3r^2 - 6r + 1 = 0$. Solving this quadratic and noting that there is only one solution $r < 1$, we obtain $r = \frac{3-\sqrt{6}}{3}$. For this value of r (one need to check that the second derivative at r is negative, but that is obvious), we obtain that $M(r) =$

Problem 4
If f is a non-constant holomoprphic function then $|f(z)|$ is strictly subharmonic.

Solution: The fact that $|f(z)|$ is subharmonic is easily seen from Cauchy's integral formula

$$f(z) = \frac{1}{2\pi} \int_0^{2\pi} f(z + re^{i\theta}) d\theta.$$

It follows that

$$|f(z)| \leq \frac{1}{2\pi} \int_0^{2\pi} |f(z + re^{i\theta})| d\theta,$$

with equality only when $f(z+re^{i\theta})$ is a constant. Therefore $|f(z)|$ is strictly subharmonic unless $f(z + re^{i\theta})$ is a constant and hence f is constant.

Problem 5
Let f be an entire function with only finitely many zeroes. Define

$$m(r) = \min_{|z|=r} f(z).$$

If f is not a polynomial then $m(r) \to 0$ as $r \to \infty$.

Solution: First of all, note that this result is not true for polynomials. Take $p(z) = z$ for example. The crucial thing about a non-polynomial entire function is that infinity is an essential singularity of any such function. Hence, the image of $f(\{z : |z| \geq R\})$ is dense in $\mathbb{C}$. In partiular, there is a sequence z_n such that $|z_n| \to \infty$ and $f(z_n) \to 0$. It is clear that $m(|z_n|) \to 0$. This of course does not solve the above problem, but I find it helpful to keep this sketch in mind.

The proof as one might expect is straightforward. The idea is to just show that the limit $\lim_{r\to\infty} m(r)$ exists. Since we already have the limit along a sequence, we will be done. To do this, let r be large so that all the zeroes of f lie in a disk of smaller radius, say $\frac{r}{2}$. There exists $\frac{r}{2} < r_n \leq r \leq r_{n+1}$. Since f does not have a zero in $\{z : r_n < |z| < r_{n+1}\}$, it follows by Minimum modulus principle that $m(r) \leq \min\{m(r_n), m(r_{n+1})\}$. This needs to be done! It is not complete.

Problem 6
Let $\Omega = \{z : |z| \leq 2\}$ and $I = [0, 1]$ be the line segment from 0 to 1.

a) Prove that if $f : \Omega \to \mathbb{C}$ is continuous and f is analytic on $\Omega \setminus I$, then f is analytic on Ω.

b) Give an example of a function which is bounded analytic on $\Omega \setminus I$ which can not be extended to an analytic function on Ω.

Solution:

a) It is standard Morera's theorem.

b) Take a countable dense subset of $[0, 1]$ and construct a Weierstrass type functions with zeroes on that dense set. Need to expand.

Problem 7
(See 1 Problem 6)

Problem 8
Let $\Omega \subseteq \mathbb{C}$ be a domain containing 0. Suppose that $U_n(z)$ is a sequence of positive harmonic functions on Ω and $\lim_{n\to\infty} U_n(0) = 0$. Then, U_n converges unifomrly (to 0) on any compact subset $K \subseteq \Omega$.

Solution: It is immediately clear from Harnack's inequality that $U_n(z)$ converges uniformly on compact subsets, say to $U(z)$. It follows that $U(z)$ is harmonic. Since $U_n(z) > 0$ it follows that $U(z) \geq 0$ for all z. Since $U(0) = 0$ it follows from Mean value property that $U(z) = 0$ for all z.

Problem 1
Evaluate the following integral:

$$\int_0^\infty \frac{dx}{1+x^n}.$$

Problem 2
Let a and b be complex numbers such that $0 < |a| < |b|$. Write down all the Taylor and Laurent series expansion of $f(z) = \frac{1}{(z-a)(z-b)}$ centered at 0.

Problem
Let G be a domain and $f_n : G \to \mathbb{C}$ be a sequence of holomorphic functions such that $f_n(z)$ converges for every $z \in G$. Suppose there is analytic function g_n such that $|g_n| \leq 1$ and $|f_n - g_n| \geq 1$. Prove that f_n converges uniformly on compact subsets of G.

Proof. Fisr note that g_n is a normal family. Passing to a subseuqence and after relabelling, we may assume that g_n converges to an analytic function g uniformly on compact subsets of G. Now, we note that the function $\frac{1}{f_n - g_n}$ is also a normal family, and it converges to $\frac{1}{f-g}$. It follows that the convergence is uniform on compact subsets of G. Let K be a compact subset of G. From our assumptions, it follows that $|f - g| \geq 1$, and therefore we can conclude $f_n - g_n$ converges to $f - g$ uniformly on K. And, hence f_n converges to f uniformly on K. □

Problem
Write down an infinite product which convreges to an entire function $f(z)$ with zeroes of order 1 at $\sqrt{n}, n = 1, 2, 3, \dots$, and no other zeroes. Prove the convergence.

Proof. Define $f(z) = \prod_{n=1}^{\infty} \left(1 - \frac{z}{\sqrt{n}}\right) \exp\left[\frac{z}{\sqrt{n}} + \frac{1}{2}\left(\frac{z}{\sqrt{n}}\right)^2\right]$. □