Real Analysis Prelims

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1 Techniques in Real Analysis

In this section, we describe some frequently used techniques in analysis and key theorems that are useful. Most theorems we list below can be found in any standard textbook for measure theory and functional analysis, but occasionally I will include some not-so-well-known theorems or results that are useful or interesting (often both).

2 Inequalities

Holder's inequality, Generalised Hölder's inequality, Minkowski's Inequality, Minkowski's integral inequality, Hausdorff-Young Inequality, Chebyshev-Markov inequality, Jensen's inequality, Convolution inequality, Parseval's identity, Hardy's inequality, Hardy-Littlewood weak estimate, Poincaré inequality, GNS inequality...

Theorem 1 (Hölder Inequality)**.** *Let $f \in L^p$ and $g \in L^q$ where $p^{-1} + q^{-1} = 1$. Then $fg \in L^1$ and $||fg||_1 \le ||f||_p ||g||_q$.*

Proof. Observe that $|ab| \le p^{-1}|a|^p + q^{-1}|b|^q$ yields that $||fg||_1 \le p^{-1}||f||_p^p + q^{-1}||g||_q^q$. Changing $f \to \lambda f$ and $g \to \lambda^{-1} g$, we obtain $||fg||_1 \le p^{-1}||f||_p^p \lambda^p + q^{-1}\lambda^{-q}||g||_q^q$. Minimizing the right side with respect to λ gives the desired inequality. $\square$

There is an important aspect of the above proof that needs to be emphasized. We begin with a weaker inequality and we observe that one side of the inequality has a symmetry/invariance property. We exploit that invariance to obtain a stronger result. In this sense, we have a class of inequality parametrized by $\lambda \in (0, \infty)$ and the Holder's inequality is an extreme point in that class. This also suggests a variational characterization of Holder's inequality and we give that below. The trick of symmetry-breaking will be discussed in more detail in the later sections.

Theorem 2. $||f||_p = \sup \{||fg||_1 : ||g||_q = 1\}$.

3 Some important results

Theorem 3 (Riesz representation theorem)**.** *Let $\mathcal{H}$ be a Hilbert space and $\ell : \mathcal{H} \to \mathbb{F}$ be a continuous linear functional on $\mathcal{H}$. Then, there exists $h \in \mathcal{H}$ such that $\ell(f) = \langle f, h \rangle$ for every $f \in \mathcal{H}$.*

Theorem 4 (Riesz separation lemma). *Let $\mathcal{H}$ be a Banach space and let let $E \subseteq \mathcal{H}$ be a proper closes subspace of $\mathcal{H}$. Then, there exists $x \in \mathcal{H}$ such that $||x|| \leq 1$ and $d(x, E) > 1/2$.*

An immediate consequence of Riesz's separation lemma is the non-existence of Lebesgue measure (translation invariant Borel regular measure) on infinite-dimensional Banach spaces. It must be remarked that the lemma only excludes the possibility of translation invariant measures that are (Radon) regular. Indeed, there are non-trivial counting measures on any vector space that are translation invariant.

Riesz's separation lemma highlights a crucial difference between finite and infinite-dimensional spaces. The following lemma (due to F. Riesz?) highlights another such difference. Recall that the closed unit ball in $\mathbb{R}^n$ is compact (Heine-Borel lemma). The following lemma essentially says that this phenomenon occurs only in the finite-dimensional settings.

Theorem 5 (Riesz?). *Let $\mathcal{H}$ be a Banach space and let $B \subseteq \mathcal{H}$ be the closed unit ball. B is compact if and only if $\mathcal{H}$ is finite-dimensional.*

Combined with Baire Category theorem an immediate corollary of the above result is that infinite-dimensional normed spaces are not σ-compact. This theorem should be contrasted with the Banach-Alaoglu theorem. Recall that any Banach space $\mathcal{H}$ has a natural weak-* topology as well and the Banach-Alaoglu theorem states that the unit ball (in norm topology) is compact in weak-* topology.

Theorem 6 (Borel-Cantelli Lemma). *Let $(\Omega, \mathcal{F}, \mu)$ be a measure space. Suppose $A_n \subseteq \Omega$ are measurable subsets such that $\sum_{n=1}^{\infty} \mu(A_n) < \infty$, then $\mu(\limsup_n A_n) = 0$.*

Proof. Since $\limsup A_n = \bigcap_{ge1} \bigcup_{k \geq n} A_k$, it follows that $\mu(\limsup A_n) \leq \sum_{k \geq n} \mu(A_k)$. □

Borel-Cantelli lemma is often stated for probability measure but it is clear from the proof that it holds for any measure. However, there is a useful converse of the Borel-Cantelli lemma that requires independence of A_n.[1] Borel-Cantelli lemma combined with some quantitative estimates obtained from Chebyshev/Markov type inequality is often used to upgrade convergence in measure to almost sure convergence.

Theorem 7 (Scheffe lemma, 1947). [2] *Let $f_n, f \in L^1(\mu)$ and $f_n \to f$ almost surely. Then, $||f_n - f||_1 \to 0$ if and only if $||f_n||_1 \to ||f||_1$.*

Sheffe's lemma has a very useful corollary in probability. Let $f_n\, d\mu$ is a sequence of probability measures. Suppose $f \in L^1(\mu)$ and let $f_n \to f$ pointwise, then $f_n d\mu \to f d\mu$ in total-variation norm. Sheffe's lemma can be obtained from the following lemma that was proved by Frigyes Riesz in 1928.

Theorem 8 (Riesz, 1928). *Let $p \geq 1$ and let $f_n, f \in L^p$ such that $f_n \to f$ almost surely. If $\lim ||f_n||_p = \lim ||f||_p$, then $||f_n - f||_p \to 0$.*[3]

The essence of the Riesz's lemma is that it provides a characterization for almost sure convergence to be an L^p convergence. In fact the proof of Riesz can be modified to obtain a criterion for convergence in measure to be convergence in L^p. We leave this as a fun exercise. An elegant proof of Riesz's lemma due to Novinger (1975) follows by observing that $2^p(|f_n|^p + |f|^p) - |f_n - f|^p \geq 0$ and using Fatou's lemma. However, we will present here the original proof of Riesz as a beautiful example of truncation argument.

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[1] There is a version of the converse of general measure spaces but we will not state that here.

[2] This result was already present in a work of Riesz 20 year before Scheffe.

[3] The case $p = 2$ follows trivially by noting that $||f_n - f||_2^2 = ||f_n||^2 + ||f||_2^2 - 2\langle f_n, f\rangle$.

Proof of Riesz's lamma. Define the following sequence of function

$$f_n^* = \begin{cases} |f_n|, & |f_n| \le |f| \\ |f| \operatorname{sgn}(f_n), & |f_n| > |f| \end{cases}.$$

The sequence of function $|f_n^*|$ is dominated by $|f|$ and $|f_n^*| \to |f|$ almost surely. A simple application of DCT gives $\lim ||f_n^*||_p = ||f||_p$ and $||f_n - f||_p \to 0$. Now observe that $||f_n - f_n^*||_p \to 0$ and therefore triangle's inequality yields that $||f_n - f||_p \le ||f_n - f_n^*||_p + ||f_n^* - f||_p \to 0$. $\square$

Theorem 9 (Weierstrass's extreme value theorem)**.** *Let Ω be a compact set let $f : \Omega \to \mathbb{R}$ be a lower semi-continuous function. Then, f achieves minimum on Ω.*

The proof is almost trivial and routine. But it is still instructive to include the proof here for the simple method used in the proof is often useful in other situations.

Proof. Consider a sequence x_n such that $f(x_n) \to \alpha := \inf_{x \in \Omega}(f(x))$. After passing to a subsequence and using the compactness of Ω, we may assume that $\lim x_n = x \in \Omega$. By definition we have $f(x) \ge \alpha$. On the other hand, by the lower semi-continuity of f, we have $\alpha := \lim(f(x_n)) \le f(\lim x_n) = f(x)$. $\square$

We'll talk more about the technique involved in this proof in later sections.

- Riesz Lemma, Riesz' theorem
- duality $L^p - L^q$, dual of ℓ_p spaces, dual of $C(X), C_0(X), C_b(X)$
- completeness of $\ell_p, L^p, C(X)$, Bergman space, Hardy space etc...
- compactness theorems: Arzela-Ascoli, Banach-Alaoglu, Montel's theorem, compactness in L^p spaces, Helly's selection, Prokhorov theorem etc.
- Extension/interpolation theorems: Hahn-Banach, Banach-Steinhaus, Riesz-Thorin, Marcinkewicz, Stein's interpolation
- Approximation/density theorems: Stone-Weierstrass theorem, Muntz-Sazs etc.
- Limit interchange theorems: DCT, MCT, Fatou's lemma, Fubini-Tonelli etc.
- Misc: Lebesgue density theorem, Lebesgue differential theorem, Radon-Nikodym, Lusin, Erogoff etc.

4 Examples and Counter-examples

Example 1 (Convergence in measure does not imply almost sure convergence)**.** *Let $\Omega = [0, 1]$ and let $f_{k,n} = 1$ on $[k2^{-n}, (k+1)2^{-n}]$ for $0 \le k \le 2^n - 1$ and 0 otherwise. The sequence $f_{k,n}$ converges in measure but not almost surely. In fact, the sequence $f_{k,n}$ does not have a pointwise limit for any $x \in [0, 1]$.*

Example 2 (Weak L^p does not imply L^p.)**.** *Recall that a function f is said to be weakly L^p, denoted as $f \in L^{p,\infty}$, if $\mu(|f| \ge t) \le Ct^{-p}$ for some constant C. In fact, $L^{p,\infty}$ is a Banach space with the norm $||f||_{p,\infty} := \inf\{C^{1/p} : \mu(|f| \ge t) \le Ct^{-p}\}$. These weak L^p spaces naturally arise in Euclidean harmonic analysis, for instance Hardy-Littlewood maximal function of an integrable* *function is in $L^{1,\infty}$. It is easily seen from Chebyshev-Markov inequality that $L^p \subseteq L^{p,\infty}$, but in this example we show that the containment is strict.*[4]

Let $f(x) = |x|^{-1/p}$ on $\mathbb{R}$. Then $f \in L^{p,\infty}$ because $m(|f| \geq t) = m(|x| < t^{-p}) = t^{-p}$. But $f \notin L^p$.[5]

5 Methods of Proof

5.1 Density or approximation argument

This is a technique which comes very often in measure theory. The typical application is to prove a statement like "property P holds for every object in class $\mathcal{C}$". For example, $\mathcal{C}$ can be class of measurable functions, class of Borel subsets or class of continuous functions on a compact set. The crucial thing is that the class $\mathcal{C}$ contains a class of simpler objects say $\mathcal{C}_0$ and there is some kind of approximation theorem or density theorem that says that everything in $\mathcal{C}$ can be generated from $\mathcal{A}$. It is usually helpful in such situation to prove that property $\mathcal{P}$ holds for $\mathcal{A}$.

5.2 Method of appropriate objects

This method is closely related to the density/approximation argument. The typical application is to prove a statement like "property P holds for every object in class $\mathcal{C}$". Here we begin by looking at the class $\mathcal{C}_0$ of all objects for which property P holds. The idea is to show that $\mathcal{C} \subseteq \mathcal{C}_0$. This is usually done in three steps: 1) show that $\mathcal{C}_0$ contains some subclass $\mathcal{C}_0$. 2) show that $\mathcal{C}_0$ is 'closed' under some set of operations. 3) finally some standard result would say that if $\mathcal{C}_0$ must be at least as big as $\mathcal{C}$. For example, consider the following problems:

1) Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a Lipschitz function. $f(E)$ is measurable for every Borel set E

2) Proof of Fubini-Tonelli theorem

5.3 Proving almost sure convergence

5.4 Method of moments

5.5 Tensor power trick

5.6 Exploiting symmetry to strengthen inequalities

5.7 Truncation trick

[4] Question to self: Is there a Hölder inequality for weak L^p functions?
[5] $L^p = L^{p,\infty}$ if and only if μ is finitely supported?

Problem 1
Suppose that every f in a closed subspace M of $L^\infty(-1,1)$ is continuous in some neighborhood of zero. Prove that there exists a fixed neighborhood of zero such that every $f \in M$ is continuous on that neighborhood.

Solution: If not, we can find a sequence of functions $f_n \in M_n$ such that $||f_n||_\infty = 1$ and f_n has a discontinuity at some x_n and $x_{n+1} < x_n/2$. Now consider the function

$$f := \sum_{n=0} \frac{f_n}{2^n}.$$

Since M is closed, it follows that $f \in M$. But it is clear that f is not continuous in any neighborhood of zero.

Problem 2
Let $T : H_1 \to H_2$ be a linear operator such that $\langle Tx, \rangle_{H_2} = \langle x, x\rangle_{H_1}$ for all $x \in H_1$. Then, show that $\langle Tx, Ty\rangle_{H_2} = \langle x, y\rangle_{H_1}$ for all $x, y \in H_1$.

Solution:

$$\langle T(x+y), T(x+y)\rangle = \langle x+y, x+y\rangle.$$

Expanding and rearranging both sides yields $\langle Tx, Ty\rangle = \langle x, y\rangle$. Note that the conclusion also holds if H_1 and H_2 are Hilbert spaces over $\mathbb{C}$.

Problem 3

Solution: The easiest way to do this is to show that $\{\nu : supp(\nu) \cap U = \emptyset\}$ is a closed set. To this end, let ν_n be a sequence of probability measures such that $supp(\nu_n) \subseteq U^c$ converging (weak-$*$) to some probability measure ν_0. We need to show that $supp(\nu_0) \subseteq U^c$. To do this, we observe that for any continuous function f with support contained inside U^c we have $0 \int f d\nu_n \to \int f d\nu_0$. This shows that $supp(\nu_0) \cap U = \emptyset$.

Problem 4
Let A and B be bounded measurable subset of $\mathbb{R}$ with positive Lebesgue measure. Let χ_A and χ_B be the characteristic function of A and B respectively.

a) Show that the convolution χ_A and χ_B are continuous functions and $\int_{\mathbb{R}} \chi_A \star \chi_B > 0$.

b) Show that $A + B$ contains a non-empty open interval.

Solution: Part (a) follows from a more general result that can be proved equally easily. Let $f \in L^\infty$ and $g \in L^1$ then $f \star g$ is continuous. To see this note that

$$\begin{aligned} |f \star g(x) - f \star g(y)| &\le \int |f(z)| g(x-z) - g(y-z)| dz \\ &\le ||f||_\infty \int |g(x-z) - g(y-z)| dz \to 0, \end{aligned}$$

as $|z-y| \to 0$. It should be pointed out that $\int |g(x-z) - g(y-z)|dz \to 0$ is not entirely trivial. The easiest way to prove it is to first prove the claim for compactly supported continuous function (which is fairly easy) and then use a density argument.

For part (b), we just note that

$$\begin{aligned}\int_R \chi_A \star \chi_B(x)dx &= \int_R \int_R \chi_A(z)\chi_B(x-z)dzdx \\ &= \mu(B)\mu(A) > 0,\end{aligned}$$

where we used Tonelli's theorem to integrate with respect to x first. Since we know $\chi_A \star \chi_B$ is continuous and non-negative, it follows that it must be positive at some point and hence positive on some open set U. We now observe that $\chi_A \star \chi_B = 0$ outside $A+B$. Therefore $U \subseteq A+B$.

Problem 5

Solution: Triangle inequality for norms shows that g_N converges in L^p. Suppose $g_N \to g$ in L^p. Convergence in measure is now a consequence of Chebyshev-Markov inequality,

$$\mu\{|g_n - g| > \epsilon\} \leq \frac{||g_n - g||_p}{\epsilon} \to 0.$$

Problem 6

Solution:

a) Let $f \in L^p(\mathbb{R}^n)$ and let $x, y \in \Omega$. Note that

$$\begin{aligned}|Kf(x) - Kf(y)| &\leq \int |k(x,s) - k(y,s)||f(s)|\, ds \\ &\leq ||k(x,\cdot) - k(y,\cdot)||_q ||f||_p,\end{aligned}$$

where the last inequality follows from Hölder's inequality. From the assumption, we have $||k(x,\cdots) - k(y,\cdots)||_q \to 0$ as $x \to y$. Therefore, we obtain $|Kf(x) - Kf(y)| \to 0$ as $x \to y$.

b) Let B be be the closed unit ball in $L^p(\mathbb{R}^n)$. Since we know that the image of K is contained in $C(\Omega)$, it suffices to show that $K(B)$ satisies the hypothesis of Arzéla-Ascoli theorem. To this end, we observe that if $||f||_p \leq 1$, then from previous observation it follows that

$$|Kf(x) - Kf(y)| \leq ||k(x,\cdot) - k(y,\cdot)||_q.$$

And, therefore $\{Kf : f \in B\}$ is equicontinuous. Also, note that $s \to ||k(s,\cdot)||_q$ is continuous from Ω to $\mathbb{R}$. Since Ω is compact, it follows that $sup\{||k(s,\cdots)||_q : s \in \Omega\} \leq M < \infty$. In particular, by Hölder's inequality we obtain

$$||Kf||_\infty \leq M,$$

for every $f \in B$. The conclusion now follows from Arzela-Ascoli theorem.

Problem 7

Solution: Standard Exercise.

Problem 8

Solution:

a) Straightforward calculation shows that

$$\begin{aligned} ||f_n||_p^p &= n^p \sum_{k=0}^{n-1} \left| \int_{k/n}^{k+1/n} f(x)dx \right|^p \\ &\leq \sum_{k=1}^{n} \int_{k/n}^{k+1/n} |f(x)|^p \, dx = ||f||_P^p, \end{aligned}$$

where the inequality in the last line follows from the Jensen's inequality.

b) Let $\epsilon > 0$ be given. Fix n large so that $|f(x) - f(y)| \leq \epsilon$ whenever $|x - y| \leq 1/n$. For $x \in [k/n, k + 1/n]$, we have

$$\begin{aligned} |f_n(x) - f(x)| &\leq n \int_{k/n}^{k+1/n} |f(y) - f(x)|dy \\ &\leq \epsilon. \end{aligned}$$

It follows that $f_n \to f$ uniformly.

c) First observe that b) implies that $f_n \to f$ in L^p if f is continuous. Now let $h \in L^p$ and let $\epsilon > 0$ be given. Let g be continuous such that $||h - g||_p < \epsilon/3$ and let n be large so that $||g - g_n||_p \leq \epsilon/3$. Observe that

$$||h - h_n||_p \leq ||h - g||_p + ||g - g_n||_p + ||g_n - h_n||_p.$$

Finally observe that $(g_n - h_n) = (g - h)_n$ and therefore from a) and b) above we obtain that $||h - h_n||_p \leq \epsilon$.

Problem 1

Solution:

Problem 2

Solution: Let $B(x, r)$ be a ball of radius r centered at x. It follows that $f(B(x, r)) \subseteq B(f(x), Lr)$. In other words, we obtain $|f(B(x, r))| \leq L^n |B(x, r)|$. It follows that $|f(A)| \leq L^n |A|$ for any A contained in the algebra generated by open balls. The conclusion now follows from the Dynkin's theorem or from the regularity of Lebesgue measure. Need to add the proof of measurability of $f(A)$.

Problem 3

Solution:

Problem 4

Solution:

Problem 5

Solution:

Problem 6

Solution:

Problem 7

Solution:

Problem 8
Let A be a closed linear operator on a Hilbert space $\mathcal{H}$.

(a) Show that A^* is bounded linear operator and is defined on all of H.

(b) Show that for every $a, b \in H$, the system of equations

$$x + A^* y = a$$
$$Ax - y = b$$

has a unique solution $x, y \in H$.

Solution:

a) We first show that the domain of A^* is all of H. To this end, fix $y \in H$ and note that $\Lambda_y : H \to \mathbb{C}$ defined by $\Lambda_y(x) = \langle Ax, y\rangle$ is a bounded linear functional. It follows from Riesz representation theorem that there exists $z \in H$ such that $\Lambda_y(x) = \langle x, z\rangle$. This proves that y is in the domain of A^*. Since y was arbitrary, our claim follows. We now show that A^* is bounded. To this end, let $y \in H$ such that $||y|| \leq 1$. Observe that

$$\begin{aligned} ||A^*y|| := \sup_{x:||x||\leq 1} |\langle x, A^*y\rangle| &= \sup_{x:||x||\leq 1} |\langle Ax, y\rangle| \\ &\leq \sup_{x:||x||\leq 1} ||Ax|| \\ &\leq ||A||. \end{aligned}$$

It follows that $\sup_{y:||y||\leq 1} ||A^*y|| \leq ||A||$. That is, A^* is bounded and its norm is bounded by $||A||$.

(b) First of all observe that $B := I + AA^*$ or $B := I + A^*A$ are invertible. To see this note that $\|Bx\|^2 = \|x\|^2 + \|A^*x\|^2 + \|AA^*x\|^2$. Therefore, B is injective ($Bx = 0 \implies x = 0$). We now show that B has a closed range and hence it must be invertible. To see that B has a closed range let $y_n = Bx_n$ be a cauchy sequence with imit $y \in H$. We need to show that $y \in Range(B)$. To this, end note that $\|x_n - x_m\| \leq \|B(x_n - x_m)\| \leq \|y_n - y_m\|$. That is, x_n is also Cauchy and hence $x_n \to x \in H$, by continuity of B it follows that $Bx = y$. Onc we know that $I + AA^*$ is invertible, it is easy to see that the solution to the above system of equations is given by $y = B^{-1}(Aa - b)$ and $x = (I + A^*A)^{-1}(a + A^*b)$.

Problem 1

Solution: Standard exercise in integration by parts and induction.

Problem 2

Solution: Define $L(x) = \lim\langle x, y_n\rangle$. It is clear that $L : H \to \mathbb{C}$ is linear. It follows from Banach-Steinhaus theorem that L is bounded and hence continuous. It follows from Riesz representation theorem that $L(x) = \langle x, y\rangle$ for some $y \in H$.

Problem 3

Solution: Assume f is continuous. Let $(x_n, f(x_n)) \in G(f)$ be a sequence that converges to $(x, y) \in X \times Y$. It follows from the continuity of f that $y = f(x)$. Therefore, $(x, y) \in G(f)$. In other words $G(f)$ is closed, and being a subset of compact set $X \times f(X)$ it is compact.

Assume that $G(f)$ is compact. Let $x_n \to x \in X$. We now show that $f(x_n) \to f(x)$. If not, then after passing to a subseuquence we can assume that $|f(x_n) - f(x)|\epsilon$ for all n. Consider the sequence $(x_n, f(x_n))$. Since $G(f)$ is compact, it follows that there is a subsequence of $(x_n, f(x_n))$ that converges to $(x, f(x))$ but this contradicts that $|f(x_n) - f(x)| > \epsilon$ for all n.

Problem 4

Solution: After translation, we may assume $f \equiv 0$. By scaling it suffices to show that the set $B = \{g : ||g|| \leq 1\}$ is not compact. To this end, consider the family of functions $f_{k,n} = \sin(2^n \pi x)$ if $x \in [k2^{-n}, (k+1)2^{-n}]$ and 0 otherwise. It is clear that $f_{n,k} \in B$ but it has no convergent subsequence because $||f_{n,k} - f_{m,j}|| = 1$ for all $(n, k) \neq (m, j)$. This completes the proof of a).

For (b), we first note that it follows from part a) that a compact set $K \in C[0, 1]$ has empty interior. It follows from the Baire's category theorem that $\bigcup K_n$ has non-empty interior for any countable collection of compact sets K_n, in particular it can't be $C[0, 1]$.

Problem 5

Solution: Standard Exercise.

Problem 6

Solution: It follows from the duality of L^1 that $h_a(x) := f(x) - f(x + a)$ is equal to 0 almost everywhere for every a. In particular, we have

$$\frac{1}{|B(\epsilon)|} \int_{B(0,\epsilon)} f(x)dx = \frac{1}{|B(\epsilon)|} \int_{B(a,\epsilon)} f(x)dx.$$

From Lebesgue differentiation theorem it follows that $f(x) = c$ almost everywhere.

Problem 7

Solution: For every $r \in (0,1)$ let α_r be the sequence obtained from the binary expansion of r. Clearly $|\alpha_r - \alpha_s| = \delta_{r,s}$ and $|\alpha_r| = 1$. Therefore, ℓ^∞ is not separable.

Problem 8

Solution: Repeatedly using the inequality for $f(t)$ and using the fact that $f(t) \geq 0$, we obtain

$$f(t) \leq a\left(1 + \sum_{i=1}^{n} \frac{b^k t^k}{k!} + \int_{x_n}^{t} \cdots \int_{0}^{x_1} f(x)dx \prod_{i=1}^{n} dx_i\right).$$

Let $|f(x)| \leq M(t)$ on $[0,t]$ then

$$\int_{x_k}^{t} \cdots \int_{0}^{x_1} f(x)dx \prod_{i=1}^{k} dx_i \leq M(t)\frac{t^n}{n!} \to 0,$$

as $n \to \infty$. It follows that

$$f(t) \leq ae^{bt}.$$

Problem 1

Solution: After passing through a subsequence, we may assume that $f_j \to f$ almost surely and in L^1. Therefore, $|f_j|^p \to |f|^p$ almost surely. Fatou's lemma now gives the part a). (It should also be noted that Fatou's lemma uses a weaker assumption that is given in the problem. In particular, it suffices to have $\liminf \int |f_n|^p < \infty$.)

To see that part b) is false consider the following counter-example. Let $p > 1$ be fixed. Let $f_{k,n} = n^{1/p}$ on $[k/n, (k+1)/n]$ and 0 otherwise. Note that $\int |f_{n,k}|^p = 1$ and $f_{k,n} \to 0$ in L^1 because $\int |f_{n,k}| = n^{1/p-1} \to 0$. Clearly $f_{k,n}$ does not converge in L^p.

Problem 2

Solution: Let $B = \{f \in L^p : ||f||_p \leq 1\}$ be the unit ball. We need to show that $\overline{T(B)}$ is compact in $C[0,1]$. If $f \in B$ then

$$(Tf)(x) \leq x^{1-1/p}||f||_p \leq 1,$$

where the first inequality follows from Hólder's inequality. Similarly,

$$|Tf(x) - Tf(y)| \leq |y-x|^{1-1/p}||f||_p \leq |y-x|^{1-1/p}.$$

It follows from Arzela-Ascoli theorem that $\overline{T(B)}$ is compact in $C[0,1]$. Therefore, T is a compact operator.

To see that the conclusion fails when $p = 1$, take $f_n = nx^{n-1} \in L^1[0,1]$. It is clear that $||f_n||_1 = 1$. Now notice that $(Tf_n)(x) = x^n$. Any subsequence of x^n does not converge in $C[0,1]$.

Problem 3

Solution: For a) take (e_j) to be any orthonormal basis for $L^2(\mathbb{R})$. It is clear that $e_j \to 0$ weakly, but $||e_j - 0||_2 = 1$.

Part b) is a simple exercise with inner products. Expanding the $||f_j - f||_2^2$ we obtain

$$||f_j - f||_2^2 = ||f_j||^2 + ||f||_2^2 - 2\langle f_j, f\rangle.$$

Since $f_j \to f$ weakly, it follows that $\langle f_j, f\rangle \to \langle f, f\rangle = ||f||^2$. The condition $\lim ||f_j||_2^2 = ||f||_2^2$ now gives the conclusion.

Problem 4

Let f be real valued Borel measurable on $[0,1]$. Show that there exists a sequence of polynomials P_k such that

$$\lim_{k\to\infty} P_k(x) = f(x) \text{ pointwise almost every } x \in [0,1].$$

Solution: Let f be Borel measureable. There exists a sequence of continuous function g_n such that $g_n \to f$ almost surely. By Weierstrass theorem we know that the polynomials are dense in $C[0,1]$. Let P_k be a sequence of polynomials such that $||g_k - P_k||_\infty \leq \frac{1}{k}$. It follows that $P_k \to f$ almost everywhere.

Problem 5
Let $\ell^2(\mathbb{N})$

Problem 6
Suppose $E \subseteq \mathbb{R}$ be measurable subset of strictly positive measure. Show that there exists a $\delta_0 > 0$ such that $\mu(E \cap (E+\delta)) > 0$ for all $0 < \delta < \delta_0$.

Solution: This is easy if E is an open interval. Let $E = (a, b)$, then $\mu(E \cap E + \delta) > 0$ for all $\delta < (b - a)$.

If E is an arbitrary set of positive measure. We can find an open interval I such that $\mu(E \cap I) \geq 9/10\mu(I) > 0$. Now choose δ_0 such that $\mu(I \cap I + \delta) > 9/10\mu(I)$ for all $\delta \leq \delta_0$. We now note that

$$\begin{aligned}\mu(E \cap E + \delta) &\geq \mu(I \cap (I + \delta)) - 2\mu(I \setminus E) \\ &= 7/10\mu(I) > 0.\end{aligned}$$

Problem 7

Solution:

Problem 8

Solution:

a) This is obvious. Since $\|S_n v\| \leq \frac{1}{n}\sum_{i=0}^{n-1}\|U^i v\| \leq \frac{1}{n}\sum_{i=0}^{n-1}\|v\| = \|v\|$. Since v is arbitrary, it proves the claim.

b) This is also obvious. If $Uv = v$ then $U^i v = v$ for all $i \geq 0$ by induction. It follows that $S_n v = \frac{1}{n}\sum_{i=0}^{n-1} U^i v = \frac{1}{n}\sum_{i=0}^{n-1} v = v$.

c) Note that

$$\begin{aligned}S_n w &= \frac{1}{n}\sum_{i=0}^{n-1}\left(U^{i+1}v - U^i v\right) \\ &= \frac{1}{n}\sum_{i=1}^{n}(U^n v - v)\ .\end{aligned}$$

It follows that $\|S_n w\| \leq \frac{1}{n}\|U^n v\| + \frac{1}{n}\|v\| \leq \frac{2}{n}\|v\|$ where the last inequality follows from the fact that $\|U^n v\| \leq \|v\|$.

d) First note that $I = ker(U - I)$ and $U - I$ is bounded linear operator, therefore I is a closed subspace. If $B^\perp = I$ then it will follow that $\mathcal{H} = \overline{B} \oplus I$. Therefore, it suffices to show that $B^\perp = I$. To this end, let $f \in I$ and $g = Uh - h \in B$, we need to show that $\langle f, g\rangle = 0$. To do this, observe that $\langle f, g\rangle = \langle f, Uh\rangle - \langle f, h\rangle = \langle Uf, Uh\rangle - \langle f, h\rangle$. The claim now follows from the observation that $\langle Uf, Uh\rangle = \langle f, h\rangle$ because $\|Uv\| = \|v\|$.

e)

Problem 1

Solution:

Problem 2

Solution:

a) We first show that $Tf \in C[0,1]$. To do this, we fix $x < y \in [0,1]$ and applying Hölder's inequality we obtain $|Tf(x) - Tf(y)| \le \int_x^y |f(\xi)| d\xi \le \|f\|_p |y-x|^{1/q}$ where $p^{-1} + q^{-1} = 1$. Therefore, Tf is continuous if $q < \infty$ or equivalently if $p > 1$. Taking $x = 0, y = 1$ and taking supremum over all function with norm $\|f\| \le 1$, we also see that T is a bounded linear operator with $\|T\| \le 1$. To show that T is compact, we show that the image of the unit ball in L^p is pre-compact. To this end, we make the following two observations. For any $f \in L_p$ such that $\|f\| \le 1$ we have

   i) $\|Tf\|_\infty \le 1$. Therefore, $\{Tf : \|f\| \le 1\}$ is bounded subset of $C[0,1]$.

   ii) $|Tf(x) - Tf(y)| \le |y-x|^{1/q}$. Therefore, $\{Tf : \|f\| \le 1\}$ is equicontinuous.

   It follows from Arzela-Ascoli theorem that $\{Tf : \|f\|_p \le 1\}$ is pre-compact.

b) Consider the sequence of functions $f_n(x) = n$ if $x \in [0, \frac{1}{n}]$ and 0 otherwise. Note that $\|f_n\|_1 = 1$ for all $n \ge 1$. It is clear that $|Tf_n(0) - Tf_n(x)| = 1$ for all $n \ge \frac{1}{x}$. In other words, Tf_n is not equicontinuous at 0. (Alternatively, one can say that $Tf_n \to h$ where $h(0) = 0$ and $h(x) = 1, x > 0$ which is not continuous.) However, it is interesting to note that $T : L^1 \to L^1$ is indeed a compact operator. Add an example of L^1 function such that Tf is no continuois.

Problem 3

Solution:

a) Let $f_n := \chi_{[n,n+1]}$. It is easy to verify that $f_n \to f \equiv 0$ weakly, but $\|f_n - f\|_2 = 1 \not\to 0$.

b) Expand the $\|f_j - f\|^2$ and we see

$$\|f_j - f\|^2 = \|f\|^2 + \|f_j\|^2 - 2\langle f_j, f\rangle .$$

   Since $f_j \to f$ weakly, we get $\langle f_j, f\rangle \to \|f\|^2$ and by assumption $\|f_j\|^2 \to \|f\|^2$. We therefore obtain that $\|f - f_j\|^2 \to 0$.

Problem 4

Solution:

Problem 5

Solution:

Problem 6

Solution: Without loss of generality assume that E has finite measure. Now let $f(x) := (\chi_{-E} \star \chi_E)(x)$. Being a convolution of an L^1 function, with an L^∞ function we know that f is continuous, moreover note that $f(0) = \mu(E) > 0$. Since f is continuous, it follows that there is an interval I containing 0 such that $f > 0$ on I. It follows that there exists $\delta_0 > 0$ such that $0 \in [-\delta_0, \delta_0] \subseteq I \subseteq E - E$. For any $\delta \leq \delta_0$ we have $\mu(E \cap E + \delta) > 0$.

Problem 7

Solution:

Problem 8

Solution:

Problem 1

Let K be the family of all non-empty compact subsets of $\mathbb{R}$. For $A, B \in K$, let

$$d(A,B) = \max\left(\sup_{y\in A}\inf_{x\in B}|x-y|, \sup_{y\in B}\inf_{x\in A}|x-y|\right)$$

Assume without proof that d is a metric on K. Prove that K with the metric d is complete.

Solution:

Problem 3

Suppose that $f_n : \mathbb{R} \to [0,1]$ for $n = 1, 2, \ldots$, and that each one of these functions is non-decreasing, that is, $f_n(x) \le f_n(y)$ if $x \le y$, for all n, x and y. Prove that there exist a function $g : \mathbb{R} \to [0,1]$, a finite or countably infinite set $A \subset \mathbb{R}$, and a subsequence f_{n_k} such that $\lim_{k\to\infty} f_{n_k}(x) = g(x)$ for all $x \in \mathbb{R}\backslash A$.

Solution:

Problem 3
Let S be a subset of $\mathbb{R}$ with strictly positive Lebesgue measure, and let $\mathbb{Q}$ denote the set of rational numbers in $\mathbb{R}$. Prove that almost every (with respect to Lebesgue measure) real number can be written as the sum of an element of S and an element of $\mathbb{Q}$.

Solution:

Problem 8
Let f be a continuous real-valued function on $[0, \infty)$ with $f(0) = 0$. Suppose that for each $y \in [0, 1]$ we have that $f(ny) \to 0$ as $n \to \infty$ through the integers. Prove that $f(x) \to 0$ as $x \to \infty$ through the reals.

Solution: Fix an $\epsilon > 0$ and consider the set

$$A_m^\epsilon := \{x \in [0, 1] : |f(kx)| \leq \epsilon, k \geq m\}.$$

Now note that $[0, 1] = \bigcup_{m\geq 1} A_m^\epsilon$. By Baire's category theorem, there exists some m_0 such that A_{m_0} has a non-empty interior. Let $(\alpha, \beta) \subseteq A_{m_0} \subseteq [0, 1]$. Therefore, we obtain that $|f(kx)| \leq \epsilon$ whenever $k \geq m_0$ and $x \in (\alpha, \beta)$. Fix $n_0 \geq \max m_0, \alpha/(\beta - \alpha)$, then $|f(y)| \leq \epsilon$ for any $y \geq n_0\alpha$. This completes the proof.

Problem 2
A metric space is said to be separable if it has a countable dense subset. Suppose μ is a finite Borel measure on a metric space (X, d). We say μ is tight if, for every $\epsilon > 0$, there exists a compact set $K_\epsilon \subseteq X$ such that $\mu(K_\epsilon) > \mu(X)(1-\epsilon)$. Show that, if X is complete and separable, every finite measure μ is tight.

Solution: Let $D = \{x_i\}_{i=1}^\infty$ be a countable dense subset of X. Then for all $m \geq 1$, we have $\bigcup_{i=1}^\infty B(x_i, 1/m) = X$. So $\mu(\bigcup_{i=1}^\infty B(x_i, 1/m)) = \mu(X) < \infty$ and thus, $\lim_{k\to\infty} \mu(\bigcup_{i=1}^k B(x_i, 1/m)) = \mu(X)$. Hence, there exists $k_m \in \mathbb{N}$, such that $\mu((\bigcup_{i=1}^{k_m} B(x_i, 1/m))^c) < \epsilon'/2^m$. Let $K = \bigcap_{m=1}^\infty (\bigcup_{i=1}^{k_m} \overline{B(x_i, 1/m)}) \subset X$. Then clearly K is closed and hence complete. Now for any $\delta > 0$, let $\{B(x, \delta)\}_{x\in K}$ be an open cover of K by δ-balls. Then there exists $1/n < \delta$. Then $K \subset \bigcup_{i=1}^{k_n} \overline{B(x_i, 1/n)} \subseteq \bigcup_{i=1}^{k_n} B(x_i, \delta)$. This implies that K is totally bounded, and hence a compact subset of X. It follows immediately that $\mu(K^c) < \epsilon'$. Let $\epsilon' = \mu(X)\epsilon$. Then $\mu(K) > \mu(X)(1-\epsilon)$. This completes the proof.

Problem 8
A subset C of a Hilbert space $(H, \|\cdot\|)$ is said to be convex if for all $x, y \in C$ and all $0 \leq t \leq 1$, the linear combination $tx + (1-t)y \in C$. Show that, given a nonempty, closed, convex subset $C \subseteq H$, any element $x \in H$ has a unique element $y \in C$ such that

$$\|x - y\| \leq \|x - z\| \quad \text{for all } z \in C.$$

Hint: Consider a minimizing sequence and argue that it is Cauchy.

Solution: Define $D = \{x - z | \ z \in C\} \subset \mathbb{H}$. Easy to check that if C is convex then so is D. Furthermore, $z \mapsto x - z$ is a homeomorphism of $\mathbb{H}$ onto itself (it is its own inverse) since $\|(x-z) - (x-y)\| = \|z - y\|$. Thus, if C is closed then so is D. Thus, it suffices to prove that any non-empty, closed, convex subset D of $\mathbb{H}$ has a unique element of the least norm. This is a standard result in elementary Hilbert space theory: Let $\delta = \inf\{\|x\| : x \in D\}$. By definition, there is a sequence of points $x_n \in D$ such that $\|x_n\| \to \delta$ as $n \to \infty$. For any $y, z \in \mathbb{H}$, we have $\|y+z\|^2 + \|y-z\|^2 = 2\|y\|^2 + 2\|z\|^2$. Applying this to $x_n/2$ and $x_m/2$, we have $\|x_n - x_m\|^2/4 = \|x_n\|^2/2 + \|x_m\|^2/2 - \|(x_n + x_m)/2\|^2 \leq \|x_n\|^2/2 + \|x_m\|^2/2 - \delta^2$, since D is convex and $(x_n + x_m)/2 \in D$. Thus, as $n, m \to \infty$, $\|x_n\|, \|x_m\| \to \delta$ and hence $\|x_n - x_m\|^2 \to 0$. Hence, x_n's form a Cauchy sequence, which converges to a limit say $x_0 \in \mathbb{H}$, by virtue of its completeness. Since D is closed, we have $x_0 \in D$. By continuity of the norm function, it follows that $\|x_0\| = \lim_{n\to\infty} \|x_n\| = \delta$. This proves the existence part of the problem.

For uniqueness, let $y, z \in D$ be such that both have norm equal to δ. Then $\|y/2 - z/2\|^2 = \|y\|^2/2 + \|z\|^2/2 - \|(y+z)/2\|^2 = \delta^2 - \|(y+z)/2\|^2 \leq 0$, since $(y+z)/2 \in D$, by its convexity. Thus, it follows that $y = z$, establishing the uniqueness.

Problem 1
Suppose A is a lebesgue measurable subset of $\mathbb{R}$ with $\lambda(A) > 0$. Show that for all $\lambda(A) > b > 0$ there is a compact set $B \subset A$ with $\lambda(B) = b$.

Solution: Given $\lambda(A) > b$. By inner regularity of Lebesgue measure, there exists compact subset $K \subset A$ such that $\lambda(A) > \lambda(K) > b$. By Heine-Borel, K is closed and bounded and thus contained in a bounded interval $[-d, d]$. Consider the function $f : [-d, d] \to \mathbb{R}$ defined by $f(x) = \lambda([-d, x] \cap K)$. For $d > y > x > -d$, $f(y) - f(x) = \lambda([x, y] \cap K) \leq \lambda([x, y]) = x - y$. Thus f is uniformly continuous function. Note that $f(-d) = 0$ and $f(d) = \lambda(K) > b > 0$. Thus, by intermediate value theorem, there exists $x_0 \in (-d, d)$ such that $f(x_0) = \lambda([-d, x_0] \cap K) = b$. Thus, $[-d, x_0] \cap K \subset K \subset A$ is the required compact subset.

Problem 3
Let $f : [0, 1] \to \mathbb{R}$. Suppose that the one-sided derivatives

$$\begin{aligned} D_-f(x) &= \lim_{h<0,h\to 0} \frac{f(x+h)-f(x)}{h} \quad (0 < x \leq 1), \\ D_+f(x) &= \lim_{h>0,h\to 0} \frac{f(x+h)-f(x)}{h} \quad (0 \leq x < 1) \end{aligned}$$

exist for all x in the indicated ranges and are bounded in absolute value by a constant $K < \infty$. Prove that the (two-sided) derivative $f'(x)$ exists for almost every $x \in (0, 1)$.

Solution:

Problem 8
Let $B(x, r) \subset \mathbb{R}^2$ denote the open disc with center x and radius r and let $S(x, r)$ be the boundary of $B(x, r)$. Let $D = B((0, 0), 1)$ be the unit open disc, and let $\mathcal{H}$ be the family of all bounded Borel measurable functions $f : D \to \mathbb{R}$. Let

$$\mathcal{A} = \left\{ f \in \mathcal{H} : f(x) = \frac{1}{2\pi r} \int_{S(x,r)} f(y) d\sigma(y) \text{ for all circles } S(x, r) \subset D \right\}$$

where $d\sigma(y)$ denotes the arc length measure on $S(x, r)$, and

$$\mathcal{B} = \left\{ f \in \mathcal{H} : f(x) = \frac{1}{\pi r^2} \int_{B(x,r)} f(y) dy \text{ for all discs } B(x, r) \subset D \right\}$$

where dy denotes 2 -dimensional Lebesgue measure. Prove that $\mathcal{A} = \mathcal{B}$. (It may be useful to show that all functions in $\mathcal{B}$ are continuous.)

Solution: The fact that $\mathcal{A} \subseteq \mathcal{B}$ is more or less obvious. Let $f \in \mathcal{A}$. Fix $x \in D$ and let $r > 0$ be such that $B(x, r) \subseteq D$. Then

$$\begin{aligned} \frac{1}{\pi r^2} \int_{B(x,r)} f(y)\, dy &= \frac{1}{\pi r^2} \int_0^r \int_{S_{x,t}} f(t\omega) d\sigma(\omega)\, dt \\ &= \frac{2}{r^2} \int_0^r f(x) t dt = f(x)\ . \end{aligned}$$

It follows that $\mathcal{A} \subseteq \mathcal{B}$. On the other hand, let $f \in \mathcal{B}$, we first show that f is continuous. To see this, observe that let $x, y \in D$ and choose $r > 0$ sufficiently small so that $B(x,r) \cup B(y,r) \subseteq D$ and observe that

$$\begin{aligned} |f(x) - f(y)| &\leq \frac{1}{\pi r^2} \int_{B(x,r)\Delta B(y,r)} |f(\xi)| \, d\xi \\ &\leq \|f\|_\infty \frac{1}{\pi r^2} \mu(B(x,r)\Delta B(y,r)). \end{aligned}$$

Since $\mu(B(x,r)\Delta B(y,r)) \to 0$ as $|x - y| \to 0$, the continuity of f follows.