Serpeienski's example of power series convergent on every point of the circle but discontinuous on circle

Raghavendra Tripathi

February 5, 2023

Abstract

The purpose of this note is to give an example of a potential series which converges at any point of its circle of convergence, but whose sum is discontinuous on this circle.

1 Introduction

Denote by P the potential series obtained by omitting the parentheses in the series

$$\sum_{n=1}^{\infty} \frac{n^2}{2^n} z^{2^n} (\alpha_n^{2^n-1} + \alpha_n^{2^n-2} z + \ldots + \alpha_n z^{2^n-1} + z^{2^n-1}), \tag{1}$$

where

$$\alpha_n = \frac{n^2 - 2 + 2n\iota}{n^2+1}. \tag{2}$$

Now we demonstrate that the series (1) converges on $|z| = 1$. It is immediate from formula (2) that

$$|\alpha_n| = 1, \tag{3}$$

and

$$|1 - \alpha_n| = \frac{2}{\sqrt{1+n^2}}. \tag{4}$$

Now let z be a complex number such that $|z| = 1$. For $|z| \neq \alpha_n$, we can write the nth term of the series (1) as

$$u_n(z) = \frac{n^2}{2^n} z^{2^n} \frac{z^{2^n} - \alpha_n^{2^n}}{z - \alpha_n},$$

which yields

$$|u_n(z)| \leq \frac{2n^2}{2^n \cdot |z - \alpha_n|}, \quad \text{for } z \neq \alpha_n, |z| = 1. \tag{5}$$

Consider the two cases:

1.1 $z \neq 1$

Suppose $|1 - z| \geq 2\delta$ for some $\delta > 0$. For $n > \frac{2}{\delta}$, we obtain $\sqrt{n^2+1} > n > \frac{2}{\delta}$, which accoridng to (4) gives $|1 - \alpha_n| < \delta$. Therefore, we obtain

$$|z - \alpha_n| = |(1 - z_n) - (1 - \alpha_n)| \geq |1 - z| - |1 - \alpha_n| > \delta,$$

which gives us that

$$|u_n(z)| < \frac{2n^2}{2^n \delta}, \quad \text{for} \quad n > \frac{2}{\delta}.$$

This shows that the series (1) converges for $|z| = 1, z \neq 1$ (since the series $\sum \frac{n^2}{2^n}$ converges).

1.2 $z = 1$

From (**??**) and (5), we find

$$|u_n(z)| \leq \frac{n^2\sqrt{n^2+1}}{2^n}, \quad \text{for} \quad n = 1, 2, 3, \ldots,$$

which yields, as we see without difficulty, that the series (1) converges for $z = 1$. The series therefore always converges for $|z| = 1$. Let us denote, for demonstration, by $P_k(z)$ the sum of the first $k-1$ terms of the series $P(z)$. Suppose k be an index > 1: There exists a natural number n (depending on k) such that

$$2^n \leq k < 2^{n+1}; \tag{6}$$

so, by denoting by $S_n(z)$ the sum of the first n terms of the series (1), we obtain

$$P_k(z) = S_{n-1}(z) + \frac{n^2 z^{2^n}}{2^n}(\alpha_n^{2^n-1} + \alpha_n^{2^n-2}z + \ldots + \alpha_n^{2^{n+1}-k-1}z^{k-2^n}),$$

from where

$$P_k(z) - S_{n-1}(z) =$$

which gives

$$|P_k(z) - S_{n-1}(z)| \leq \frac{2n^2}{2^n \cdot |z - \alpha_n|}, \quad \text{for} \quad z \neq \alpha_n, |z| = 1. \tag{7}$$

Suppose $z \neq 1$. For $k > 2^{2/\delta+1}$, let $\delta = \frac{1}{2}|1 - z|$. This results

$$|P_k(z) - S_{n-1}(z)| \leq \frac{2n^2}{2^n\delta}, \quad \text{for} \quad k > 2^{2/\delta+1}. \tag{8}$$

As $n \to \infty$ with k, it follows that sequence of partial sums $P_k(z)$ converges. For $z = 1$, the inequality (7) and (4) gives that

$$|P_k(z) - S_{n-1}(z)| \leq \frac{n^2\sqrt{n^2+1}}{2^n},$$

from which it follows that $P_k(1)$ converges. Thus we have demonstrated that the series $P(z)$ converges for $|z| = 1$. Let k be any natural number and let us calculate $P(\alpha)$. It is evidently sufficient to evaluate the series (1) at $z = \alpha_k$. Note that for any two natural number k, n we have

$$|\alpha_k - \alpha_n| = \frac{2|k-n|}{\sqrt{((k^2+1)(n^2+1)}},$$

and therefore

$$|\alpha_k - \alpha_n| \geq \frac{2}{\sqrt{((k^2+1)(n^2+1)}} \quad \text{for} \quad n \neq k.$$

Thus, we obtain from (5) that

$$|u_n(\alpha_k)| \leq \frac{n^2}{2^n}\sqrt{(k^2+1)(n^2+1)} \quad \text{for} \quad k \neq n. \tag{9}$$

This gives

$$\left|\sum_{n=1}^{k-1} u_n\alpha_k + \sum_{n=k+1}^{\infty} u_n(\alpha_k)\right| < \sqrt{k^2+1}\sum_{n=1}^{\infty}\frac{n^2\sqrt{n^2+1}}{2^n}. \tag{10}$$

Suppose $\sum_{n=1}^{\infty}\frac{n^2\sqrt{n^2+1}}{2^n} = A$ (It is obviously a positive finite number). We will have according to (10)

$$|P(\alpha_k) - u_k(\alpha_k)| < A\sqrt{k^2+1} < A(k+1). \tag{11}$$

But we evidently have

$$u_k(\alpha_k) = k^2\alpha_k^{2^{k+1}-1},$$

from which it follows that

$$|u_k(\alpha_k) = k^2|. \tag{12}$$

From inequalities (11) and (12) we obtain

$$|P(\alpha_k)| \geq |u_k(\alpha_k)| - |u_k(\alpha_k) - P(\alpha_k)| > k^2 - A(k+1),$$

which shows that by taking k sufficiently large, we can make $P(\alpha_k)$ sufficiently large. The function $P(z)$ therefore is not bounded on $|z| = 1$(around $z = 1$): therefore it is not continuous on $|z| = 1$ (being discontinuous at $z = 1$).

References