Beyond Ulam-Hammersley problem

Raghavendra Tripathi

Department of Mathematics
University of Washington, Seattle

Introduction

Theorem (Erdös-Szekeres, 1935 (A combinatorial problem in geometry))

Let $a_1, \ldots, a_{mn+1}$ be a sequence of distinct real numbers. Then there exists an increasing subsequence of length m or a decreasing subsequence of length n (or both).

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Ulam, 1935, Hammersley 1970

Let $\sigma_n \in S_n$ be a random permutation and let L_n be the length of longest increasing subsequence in σ_n. How does $L(\sigma_n)$ behave as $n \to \infty$?

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From Erdös-Szekeres theorem one obtains

$$E(L_n) = E\frac{L_n + D_n}{2} \geq E(\sqrt{L_n D_n}) \geq \sqrt{n}.$$

As a consequence, we also get

$$\liminf \frac{E(L_n)}{\sqrt{n}} \geq 1.$$

Towards accurate asymptotic

Upper bound

As $n \to \infty$, we have

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Moreover, we have

$$\frac{L(\sigma_n)}{\sqrt{n}} \to c_2,$$

in measure/probability.

More generally...

Bollabaás-Winkler, 1988. (Longest Chain amaong random points in Euclidean space)

Let n points be chosen uniformly at random from unit cube $[0, 1]^d$. Let L_n^d be the length of longest chain. Then,

$$\lim \frac{L_n^d}{n^{1/d}} \to c_d,$$

for some $1 < c_k < e$.

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Questions

- Is c_d monotonic?
- Non-trivial lower bound on c_d.
- precise value of c_d? Any guess?

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- Easy Direction: $c_2 \leq 2$.
- Show that $E(L_n) - E(L_{n-1}) \leq \frac{1}{\sqrt{n}}$.

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- Show that $E(L_n) - E(L_{n-1}) \leq \frac{1}{\sqrt{n}}$.
- $c_2 \geq 2$ is extremely involved.

Theorem (Logan-Shepp)

- RSK correspondence gives a pair of Young-tableau.
- Under this correspondence we push forward the uniform measure on S_n to the space of Young diagrams.
- Young diagram corresponds to irreducible representations of S_n of maximal dimension.