Random Matrices, $\beta-$ensembles, and Universality

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Random Matrix Theory

- A random matrix (ensemble) $A_N = (A_N(i,j))_{1 \leq ij \leq N}$, where $A_N(i,j) \sim$ some distribution.

Examples

- Let $B_N(i,j) = \pm 1$ with probability $1/2$.
- Let $X_{i,j} \sim \mathbb{C}\mathcal{N}(0,1)$ be i.i.d. Set $A_N = (X_{i,j})_{1 \leq i,j \leq N}$.
- Let $G_N = \frac{1}{\sqrt{2}}(A_N + A_N^*)$. (GUE)
- Adjacency and Laplacian of Random Graphs

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Questions?

- Matrix Norm
- Determinant
- Joint Density of Eigenvalues/singular-values
- Eigenvectors and so on.

Definitions, Notations and Plan!

- For a matrix A with eigenvalues $\lambda_1, \ldots, \lambda_N$, the empirical spectral distribution, $ESD(A)$, is the probability measure defined as

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Questions!

- (Global Question) Limiting Behaviour of ESD of a (normalized) random matrix ensemble.
- (Local Question) Local statistics of eigenvalues, for instance, the spacing between consecutive eigenvalues!

In the beginning...

Wishart Ensemble

- Let $X = X_{N,M}, N \leq M$ be a matrix with i.i.d. complex Gaussian entries. Let $W_N = XX^*$.
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The Generalized Product Moment Distribution in Samples from Normal Multivariate Population, John Wishart, 1928.

$$f(\lambda_1, \ldots, \lambda_N) \sim \Delta(\lambda)^2 \prod_i \lambda_i^{m-n} e^{-\lambda_i},$$

where $\Delta(\lambda) = \prod_{i<j} |\lambda_i - \lambda_j|$.

continued...

Wigner's semicircular law

- Let A_N be a GUE matrix i.e. $A_N(i,j) \sim \mathbb{C}N(0,1), i < j$ and $A_N(i,i) \sim N(0,1)$.

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On the distribution of roots of certain symmetric matrices, Wigner, 1957

$$ESD_{\frac{A_N}{\sqrt{N}}} \implies \mu_{sc},$$

where $\frac{d\mu_{sc}}{dx} = \frac{1}{2\pi}\sqrt{(4-x^2)_+}$.

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- Wigner first proved the result for sign matrices.
- Wigner used the method of moments.
- (Universality Principle) The limiting ESD should be independent of the distribution of entries.

Ginibre Formula

Statistical Ensembles of Complex, Quaternion, and Real Matrices, Jean Ginibre, 1965

Let A_N be symmetric/Hermitian matrix with i.i.d. (real/complex) Gaussian entries. Let $\lambda_1, \ldots, \lambda_N$ be the eigenvalues of A_N, then the joint density of eigenvalues is

$$\propto \Delta(\lambda)^\beta \exp(-\beta \sum_i |\lambda_i|^2/2),$$

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- The same formula (with $\beta = 2$) holds if the entries are complex Gaussian but the matrix is not necessarily Hermitian.
- Circular Law for non-Hermitian, complex Gaussian matrix with i.i.d. entries.
- *Random matrices: universality of ESDs and the circular law*, T. Tao, V. Vu, 2010.

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$$T_{2,N} = \begin{pmatrix} N(0,1) & \frac{1}{\sqrt{2}}\chi_{2(n-1)} & 0 & \dots & 0 \\ \frac{1}{\sqrt{2}}\chi_{2(n-1)} & N(0,1) & \frac{1}{\sqrt{2}}\chi_{2(n-2)} & \dots & 0 \\ 0 & \frac{1}{\sqrt{2}}\chi_{2(n-2)} & N(0,1) & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \frac{1}{\sqrt{2}}\chi_2 \\ 0 & & & \frac{1}{\sqrt{2}}\chi_2 & N(0,1) \end{pmatrix}.$$

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- Trotter used the tridigonal form to compute the joint density of eigenvalues of *GUE* as well as the limiting *ESD*.

$$T_{\beta,N} = \begin{pmatrix} N & \frac{1}{\sqrt{\beta}}\chi_{\beta(n-1)} & 0 & \cdots & 0 \\ \frac{1}{\sqrt{\beta}}\chi_{\beta(n-1)} & N & \frac{1}{\sqrt{\beta}}\chi_{\beta(n-2)} & \cdots & 0 \\ 0 & \frac{1}{\sqrt{2}}\chi_{\beta(n-2)} & N & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \frac{1}{\sqrt{2}}\chi_{\beta} \\ 0 & & & \frac{1}{\sqrt{\beta}}\chi_{\beta} & N \end{pmatrix}.$$

Matrix Models for Beta Ensembles, Dumitriu and Edelman, 2002

$$T_{\beta,N} = \begin{pmatrix} N & \frac{1}{\sqrt{\beta}}\chi_{\beta(n-1)} & 0 & \cdots & 0 \\ \frac{1}{\sqrt{\beta}}\chi_{\beta(n-1)} & N & \frac{1}{\sqrt{\beta}}\chi_{\beta(n-2)} & \cdots & 0 \\ 0 & \frac{1}{\sqrt{2}}\chi_{\beta(n-2)} & N & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \frac{1}{\sqrt{2}}\chi_{\beta} \\ 0 & & & \frac{1}{\sqrt{\beta}}\chi_{\beta} & N \end{pmatrix}.$$

Joint density of the eigenvalues of $\frac{1}{\sqrt{N}} T_{\beta,N}$ is

$$f_{\beta,N} =\propto \Delta(\lambda)^{\beta} \exp\left(-\beta N \sum_i \frac{\lambda_i^2}{2}\right).$$

- In subsequent work by various authors the semicircle law was established for the ESD of $\frac{1}{\sqrt{N}} T_{\beta,N}$.

General $\beta-$Ensemble

- A β-ensemble is a probability measure, $dP^V_{N,\beta}$, on $\mathbb{R}^n$ with density

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- (Dumitriu-Edelman) $P^V_{N,\beta}$ for $\beta > 0$, and $V(x) = x^2/2$, is the eigenvalue density of $\frac{1}{\sqrt{N}} T_{\beta,N}$.
- For $\beta = 1, 2, 4$, there are matrix models with symmetric/Hermitian/symplectic matrices whose eigenvalue density is given by $P^V_{N,\beta}$.

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- For $\beta = 1, 2, 4$, there are matrix models with symmetric/Hermitian/symplectic matrices whose eigenvalue density is given by $P^V_{N,\beta}$.
- (Krishnapur, Rider, Virag 2013) There is a tridigonal matrix model–with dependent entries–that realizes the general β-ensemble as the eigenvalue density.

β-ensemble continued

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- For uniformly convex V, the equilibrium measure has the form $\frac{d\mu}{dx} = S(x)\sqrt{(b-x)(x-a)_+}$ with $S(x) \geq c > 0$ on $[a, b]$.
- Recall Wigner's semicircle law: The ESD converges to the semicircular law with density $\frac{1}{2\pi}\sqrt{4-x^2}$.

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Finer Questions: Fluctuations

- The largest eigenvalue of GUE is close to 2. How does it fluctuate around 2?
- What is the correct scale of fluctuation?
- How does the largest particle under β-ensemble fluctuate around the edge of μ_V?
- What does spacing between two particles look like?

Edge Universality!

$$|\{j : \lambda_j \geq 2 - \epsilon\}| \approx N \int_{2-\epsilon}^{2} d\mu_{sc}(x) \approx cN\epsilon^{3/2}.$$

- To get $O(1)$ number of particles near 2, we should zoom at $N^{-2/3}$.
- There are local semicircle law that makes this heuristic rigorous.

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Level spacing distributions and the Airy kernel, 1994, Tracy, Widom

$$\lim_{N\to\infty} P(\lambda_N \geq 2 + \frac{x}{N^{2/3}}) = F_2(x).$$

Universality!

Fluctuation of the largest particle under β-ensemble follows Tracy-Widom distribution!

(Incomplete) History at the Edge!

- Tracy-Widom (1994) discovered Tracy-Widom distribution for the fluctuation of largest eigenvalue of random Hermitian matrix.
- Soshnikov (1999) proved Tracy-Widom law for general Wigner matrices assuming symmetric distributions.
- Tao-Vu (2010) with 4-moment assmuption
- Krishnapur, Rider, Virag (2013) For $\beta \geq 0$, V convex.
- Bourgade, Erdös, Yau (2014) $\beta \geq 1, V \in C^4$.
- Shcherbina, $\beta > 0$, V analytic (multi-cut case included).
- Bekerman, Figalli, Guionnet (2015) $\beta > 0$, V non-critical.

Bulk Universality

- In the 'bulk of the spectrum', typical spacing is N^{-1}, that is, $\lambda_{i+1} - \lambda_i \approx 1/N$.
- Fix $u \in (-2, 2)$. Given $x, y \in \mathbb{R}$, we ask what is the Probability that there is an eigenvalue at $u + \frac{x}{N\mu_{sc}(u)}$ and an eigenvalue at $u + \frac{y}{N\mu_{sc}(u)}$?
- More generally, we want to understand the distribution of $N(\lambda_{i+1} - \lambda_i)$.

Bulk Universality

Gaudin-Mehta

$$P\left(\text{no eigenvalues in } [u+\frac{x}{N\mu_{sc}(u)}, u+\frac{y}{N\mu_{sc}(u)}]\right) \to 1-\left(\frac{\sin(x-y)}{x-y}\right)^2.$$

- More generally, the $N(\lambda_i - \lambda_{i+k_1}), \ldots, N(\lambda_{i+k_n} - \lambda_{i+k_{n-1}})$ has correlation kernel given by sine-kernel law.
- For $\beta > 0$, and $V(x) = x^2/2$, these correlation have explicit description in terms of Stochastic Operators.
- Universality in this context means that for a fixed $\beta > 0$, the correlations are independent of V.

Universality in Bulk

$$\int f(N(\lambda_{i+1} - \lambda_i))dP^V_{N,\beta} = \int f(N(\lambda_{i+1} - \lambda_i))dP^G_{N,\beta} + o_N(1),$$

where $G(x) = x^2/2$.

Universality at edge

$$\int f(N^{2/3}(\lambda_N - b_V))dP^V_{N,\beta} = \int f(N^{2/3}(\lambda_N - 2))dP^G_{n,\beta} + o_N(1).$$

where $G(x) = x^2/2$.

Transportation Approach to Universality!

Transport Maps for β-matrix models and Universality, 2015, Bekerman, Figalli, Guionnet

- Construct a transport map $T_N \sharp P^V_{\beta,N} \approx P^G_{\beta,N}$.
- Taylor expansion of $T^i_N = T_0(\lambda_i) + \frac{1}{N} X^i_N(\lambda) + \frac{1}{N^2} Y^i_N(\lambda)$.
- Good Estimates on norms of the maps T_0, X^i_N, Y^i_N.

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Theorem 1.5 (Universality)

Let V be smooth and let $G(x) = x^2/2$. For any Lipschitz function $f : \mathbb{R} \to \mathbb{R}$ with support in $[-M, M]$, we have

$$\left| \int f(N^{2/3} T_0'(2)(\lambda_N - 2)) dP^G_N - \int f(N^{2/3}(\lambda_N - a_V)) dP^V_N \right| \leq C \frac{(\log N)^3}{N^{1/3}}.$$

Thank You!